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Geometric Games A Brief History of the Not so Regular Solids

I. COSMOLOGY, THEOLOGY AND MATHEMATICS

1. Ancient Roots

In 1936 French archaeologists found a series of mathematical tablets at Susa, some 200 miles east of Babylon. These Babylonian tablets, dating c. 1800-1600 B.C. contain among the earliest known computations for regular polygons, triangle, square, pentagon, hexagon and heptagon.1 The earliest known semi-regular solids also came from Babylon and appear to have been used in connection with weights and measures2 (fig. 1.2-4). Their use remained practical. Neither the Babylonians nor the Egyptians developed mathematical or scientific theories about such solids.

In nature approximations of the cube and octahedron are found in pyrite crystals (FeS2) which occur in iron-ore deposits. In Switzerland and Northern Italy approximations of the icosahedron and dodecahedron are also found in such crystals in the valleys of the alps, particularly Traversella and Brosso, leading to Piedmont.3 Aside from the island of Elba this is the only region in the world where such crystals occur (cf. fig 1.1). It is significant, therefore, that interest in the twelve sided dodecahedron and twenty sided icosahedron developed in this part of Northern Italy during the iron age (c. 900-500 B.C.). At least 28 dodecahedra from this period have been recorded in museums.4 They were used as weights. Their sides were covered with number symbolism which may have come from Babylon, possibly in Egypt and/or the Phoenicians.5 The Etruscans and Celts also endowed these solids with religious symbolism which Pythagoras of Samos adopted when he came to Italy sometime between 540 and 520 B.C.6

All too little is known of his precise contribution. One tradition claims that Pythagoras studied only the cube, pyramid and dodecahedron and that Theaetetus subsequently studied the octahedon and dodecahedon. Another tradition claims that Pythagoras gave each of the five regular solids a symbolic meaning, associating the six sided cube or hexahedron with earth, the four sided pyramid or tetrahedron with fire, the eight sided octahedron with air, the twenty-sided icosahedron with water and the twelve sided dodecahedron with the universe, or the atom of the all embracing ether.7 There is evidence that he studied the construction of these regular solids in terms of triangles.8

The case of the dodecahedron is of particular interest. Each of its twelve sides is a regular pentagon. If lines are drawn to its consecutive corners a star pentagon is produced. The Pythagoreans called this star pentagon Health, and used it as a symbol of recognition for members of their school. Starting from this symbol the Pythagoreans could construct a pentagon using an isosceles triangle having each of its base angles double the verticle angle. The construction of this triangle involved the problem of dividing a line so that the rectangle contained by the whole and one of its parts is equal to the square on the other part. This is also known as the problem of the extreme and mean ratio or the problem of the golden section.9 In simple terms, the idea of squaring the sides of triangles, familiar from the Pythagorean theorem which we all learned at school, appears to have been linked with questions of using triangles to make pentagons in constructing dodecahedrons.

Pythagoras, who was referred to simply as HIM by members of his school did not write down his ideas. Probably the first to do so was his follower, Hippasus, who wrote a mathematical treatment of the dodecahedron involving its inscription in a sphere10 but, the story goes, then perished by shipwreck, for not acknowledging that everything belonged to HIM. Plato (428-348 B.C.) was more successful. He built on the Pythagorean associations in his *Timaeus*, claiming that fire (pyramid), air (octahedron) and water (icosahedron) were composed of scalene triangles and could be transformed into one another. Earth (cube), he claimed, was composed of isosceles triangles. The fifth construction (dodecahedron) "which the god used for arranging the constellations on the whole heaven"11, Plato did not explain. Indeed he seems to have added nothing essentially new to the discussion. However, thanks to the enormous popularity, which the *Timaeus* subsequently enjoyed, the five regular solids are often referred to as the five Platonic solids.

Meanwhile the mathematician Theaethetus (fl. 380 B.C.) had written the first systematic treatise on all the regular solids.12 Some sixty years later, Aristaeus, the elder (fl. 320 B.C.) wrote a *Comparison of the five regular solids* in which he proved that "the same circle circumscribes both the pentagon of the dodecahedron and triangle of the icosahedron when both are inscribed in the same sphere."13 This work was one of the starting points for Euclid (fl. 300 B.C.) who dealt with these problems much more systematically in his *Elements*. In Book II Euclid dealt with the division of a straight line in extreme and mean ratio, on which the construction of the regular pentagon depends. In Book IV he gave a theoretical construction of the regular pentagon, probably on the basis of Pythagorean sources. Euclid, however, set the problem in a much larger framework. Book IV dealt systematically with polygonal figures in a plane, i.e. two-dimensionally. Euclid showed how to 1) inscribe a triangle, square, pentagon and hexagon in a circle, 2) circumscribe these around a circle and, in turn, 3) to circumscribe a circle around these forms. In the final proposition of this book he showed how to inscribe a fifteen-sided figure in a circle. In Book XII he described the construction of a 72 sided figure. In Book XIII, having pursued problems of dividing lines into extreme and mean ratios, Euclid explained how to inscribe a pyramid (tetrahedron), octahedron, cube (hexahedron), icosahedron and dodecahedron within a sphere, ending his book by comparing the relative sizes of the sides of these solids with one another. Theoretically, Euclid described the construction of three-dimensional versions of the five regular solids. However the diagrams which have come down to us are strikingly lacking in three dimensional qualities (fig. 2.1-5). Those of Pappus14 (fl. 340 A.D.) were more convincing (fig. 3.1-5). Yet it was not until the fifteenth century that Piero della Francesca (fig. 6.4-5) and then Leonardo da Vinci (fig.8-11) produced fully three dimensional versions of the five regular solids.

The fact that Euclid ended his book with the construction of the so-called Platonic figures led later commentators such as Proclus (c. 450 A.D.) to claim that the whole argument of the Elements was concerned with the cosmic figures.15 This was an exaggeration since the Elements provided a foundation for the study of geometry in general. Yet it points to important links between cosmology and mathematics. After Euclid, Apollonius of Perga (c. 262-180 B.C.) wrote a Comparison of the

dodecahedron to the icosahedron showing their ratio to one another when inscribed within a single sphere.16 Not satisfied with this account, Basilides of Tyre and the father of Hypsicles made emendations.17 Hypsicles, in turn, wrote his own treatise on the subject in which he compared the sides, surfaces and contents of the cube, dodecahedron and icosahedron.18 This became the so-called Book XIV of the *Elements* which was often ascribed to Euclid himself.
Not everyone in Antiquity was happy with this abstract mathematical system of cosmology. Plato's best student, for instance, rejected it outright. Aristotle was committed to showing that nature could not have a vacuum. To accept the existence of regular polygonal shapes meant that there would be spaces between them. Hence he attacked Plato's cosmology on practical grounds:

> In general the attempt to give a shape to each of the simple bodies (i.e. elements) is unsound for the reason, first, that they will not succeed in filling the whole. It is agreed that there are only three plane figures which can fill a space, the triangle, the square and the hexagon, and only two solids, the pyramid and the cube. But the theory needs more than these because the elements which it recognizes are more in number.... From what has been said it is clear that the difference of the elements does not depend upon their shape.19

Such good common sense did not suffice, however, to do away with the Pythagorean ideas which Plato had adopted. The matter continued to be debated. Even so there is evidence that the practical tradition gained importance. For example, Archimedes (287-212 B.C.), the one who yelled Eureka when he discovered the principle of specific gravity in his bathtub, studied truncated versions of the regular solids and thus discovered the thirteen semi-regular shapes which are today remembered as Archimedeian solids One of earliest systematic records of these is from a manuscript now in Trieste (fig 4-6, cf. Appendix 1).20 Practical concerns with the measurement of regular shapes also developed. Hero of Alexandria (fl. 150 A.D.), for instance, measured the relative sizes of the icosahedron and dodecahedron in his *Metrics*.21 So too did Pappus of Alexandria (fl. 340 a.D.).22

2. Mediaeval Developments

Practical uses of these regular shapes probably go back to earliest times. There is evidence that they were sometimes used in games of dice.23 In Antiquity glass and bronze jewels were made in the form of a cube-octahedron (fig. 1.5) and other semi-regular solids.24 This practice continued in the Middle Ages as is attested by their frequent occurrence in graves in Hungary and Northern Europe particularly in the fifth century A.D.25

Euclid was not forgotten. For instance, Isodorus of Miletus (fl. 532), the architect of Hagia Sophia in Constantinople (i.e. Istanbul) and one of his students, added a so-called Book XV of the *Elements*, which dealt with further problems relating to the regular solids.36 In the Arabic tradition, Ishaq b. Hunain (d. 901 A.D.), in his translation of the *Elements*, improved by Thabit b. Qurra (d. 910 A.D.), included Books XIV (by Hypsicles) and XV (by Isidorus) as if they had been written by Euclid. In the preface Ishaq explained that he had given his own method of inscribing the spheres in the five regular solids and developed the solution of inscribing any one of the solids in any other, noting those cases where this could not be done.27 In the twelfth century when Gerard of Cremona translated the *Elements* back into Latin he too assumed that Euclid had written all fifteen books.28 So too did Campanus of Novara in the thirteenth century when he made his translation of the *Elements*.29 To late Mediaeval scholars it thus appeared as if Euclid had devoted three books of his *Elements* to regular solids, and since the last of these dealt with cosmology and metaphysics, it seemed as if Euclid was concerned with much more than arithmetical proportions and geometrical features. His fascination with regular solids offered a key to Nature's regularity, the structure of the elements and the cosmos itself.

Two factors greatly increased the significance of this interpretation. The original Greek term for geometry had literally meant "measurement of the earth." Notwithstanding emphasis on practical applications by the Romans, ancient geometry remained largely an intellectual exercise involving abstract figures from a world of ideas. The Christian tradition changed this. It began from a premise of what Auerbach30 has called "creatural realism," that the natural world is real, because God created it. Hence when Boethius31 (480-524) revived the notion of geometry as a measurement of the earth, it gradually acquired an entirely new meaning. For the earth was no longer a poor imitation of a world of ideas. It was a testament to God's creation and geometry was no longer a purely intellectual exercise. It involved practical comprehension of the physical universe. The Arabic tradition, particularly the strands that came to the West helped to reinforce this approach.32

Meanwhile the metaphysical interpretation of Euclid's geometry and Plato's cosmology had also been integrated within the Christian tradition, such that God himself was seen as the Divine Geometer33 and knowledge of geometry was now a means of understanding God. So practical and intellectual knowledge became interdependent and both were linked with religion. Knowing more helped one to believe more. This approach was already firmly established by the eleventh century. From the twelfth century onwards as translation of ancient texts both from the original Greek and via Arabic versions expanded into a systematic venture, all this became more significant. Cumulative dimensions of knowledge became important. For instance, Aristotle's objections to Plato's cosmology had not been forgotten. At Cordoba, the Arabic scholar Averroes (1126-1198) wrote a long commentary on the relevant passages in Aristotle's On the Heavens. Two and a half centuries later, Aristotle's passage and Averroes' commentary in turn provoked Regiomontanus to write a treatise, which set the stage for our story.

3. Regiomontanus

Regiomontanus, whose real name was Johannes Müller, was a rather amazing figure.34 He studied with Peurbach, professor of astronomy at Vienna, who had the greatest collection of scientific instruments at the time. When Cardinal Bessarion, who commuted between Rome and Venice was trying to arrange a first edition of Ptolemy's Geography he was unable to find anyone in Italy. So he went to Peurbach. When Peurbach died Regiomontanus took over. He lectured at Padua but soon moved to Nürnberg to start the world's first publishing press for scientific books. He was one of the greatest astronomers of his day, was a pioneer in trigonometry and much involved in the reform of the Gregorian calender which, it is rumoured, led him to be poisoned in Rome at the age of 40. Regiomontanus is of special interest to us because he wrote a treatise *On the Five Equilateral Bodies, Commonly Called Regular, Namely, Which of Them will Fill a Natural Place and Which of Them do not*. *Against the Commentator on Aristotle, Averroes*.35 Regiomontanus was concerned with much more than the construction of the five regular solids. He wished to demonstrate their systematic transformation from one into another. For instance, he described how to change a cube into a tetrahedron, an octahedron and a dodecahedron. He then measured these bodies.36 Next he described how one could increase the size of a cube using square roots and cube roots. He ended the chapter by demonstrating that twelve cubes did not circumscribe a thirteenth and that twelve tetrahedrons (pyramids) did not fill up a space entirely. In the next two chapters he discussed the relation of diameter to circumference in a circle and the use of these properties in transformations into circles of different sizes. A further chapter was addressed to the volume and area of circles. This was based specifically on Archimedes' work *On the Sphere and Cylinde*r. Chapters on the measurement of irregular bodies and binomials followed. Regiomontanus went on to explain how a systematic development of the regular solids could lead to an "unlimited" number of regular irregular (i.e. semi-regular) solids.37 In his final chapter he proposed how this could be done methodically.

The original manuscript is lost so we know nothing specific about its illustrations. What we know of the text is largely because Regiomontanus summarized its arguments in another work.38 From an important history of Nürnberg mathematicians in the early eighteenth century we also know that its themes were still familiar in Nürnberg at that time.39 Since Regiomontanus lectured, worked and died in Italy it is very possible that he took the manuscript with him and that mathematicians there became aware of his work either directly or indirectly. This would account for parallels between his work and that of Piero della Francesca. In any case Regiomontanus' development of Euclid's geometry in connection with regular solids and cosmology helped to set the stage for Piero della Francesca (fig. 7.1), Pacioli, Jamnitzer, and others.

4. Piero Della Francesca

In Italy one of the key figures in these developments was Piero della Francesca. Born in San Sepolcro sometime around 1410, Piero studied painting with Domenico Veneziano in Florence and became one of the great painters of the Renaissance. Today he is most famous for his fresco cycle showing the *Legend of the True Cross* (Arezzo), his *Brera Altar* (Milan), Baptism (London, National Gallery), *Flagellation* (Urbino) and *Resurrection* (San Sepolcro. He was also involved in inlaid wood (intarsia) with architectural scenes and his name is frequently associated with those three famous views of idealized cities, the Baltimore, Berlin and Urbino panels.

Characteristic of his work was a mathematical rigour and clarity. This reflected his profound interest in mathematics, on which he wrote three treatises dedicated to the Duke of Urbino. The earliest of these was a *Treatise on the Abacus*. This stood firmly in a tradition that went back to the 1220's when Fibonacci--as in Fibonacci numbers--went to North Africa, studied algebra and practical geometry with the Arabs and incorporated their rules in his *Book of the Abacus*. This had served as a starting point for an abacus school, which eventually became the Renaissance version of a business school. Leonardo da Vinci, for instance, learned his basic mathematics at one of these schools. Piero's *Treatise of the Abacus* contained practical problems such as interest rates and measurement of the volume of wine barrels. It also dealt with measurement of the regular solids, a theme that Piero pursued in his *Booklet on the Five Regular Bodies*. Part one dealt with two dimensional figures: triangles, squares, pentagons, hexagons, octagons and circles; part two, with measurement of the five regular bodies contained in a sphere (fig. 6.3-5). Part three dealt with measurement of one regular body placed within another. As Daly Davis40 has shown, parts one and two were largely based on Piero's earlier *Treatise on the abacus*.

Part three, by contrast, was based on Book XV of the *Elements*, but rearranged so that the solids in which they were contained, beginning with the tetrahedron and ending with the icosahedron were in order of complexity. In part four of his Booklet, Piero cited the work of Archimedes (287-212 B.C.) and also described five of the thirteen semi-regular bodies which history has remembered as the Archimedeian solids.

More was involved than a simple revival of ancient mathematics. Euclid, for instance, had dealt with the five regular solids as a construction problem using square roots to determine the relative lengths of their respective sides, but appears to have had no interest in either their physical reconstruction or their representation in three dimensional terms. Piero della Francesca, by contrast, was concerned with representing both the Euclidean and Archimedeian solids in spatial terms. Piero was of course working in a tradition. A generation earlier Leon Battista Alberti had written *On Mathematical Games41* in which he had dealt with problems of geometrical transformation such as quadrature of the circle and perspectival transformations of shape. Alberti had also written On Painting, the first extant treatise on perspective. Piero, in turn, wrote his third treatise, *On the Perspective of Painting* (c. 1478-1482). Ironically, this milestone in the conquest of visual space was finished after he had gone blind. In this work Piero demonstrated the perspectival foreshortening of two dimensional polygons, namely, a triangle, square, pentagon, hexagon, octagon and a sixteen sided figure, as well as a three dimensional cube.42 Piero also described the geometrical transformation of a three dimensional sphere into an egg so that one could draw an egg which appeared as a sphere when viewed from a given point.43 Here he was codifying a principle of trick perspective or anamorphosis which he had used in his *Brera Altar* (fig. 7.2) .44

Piero's egg offers a beautiful example of Renaissance symbolism. Ostrich eggs filled with perfumed salts were used as deodorants over doorways where persons took off their shoes in the mosques of Constantinople. If you go to the Blue Mosque in Istanbul you can still see them today. Piero, working about two decades after the fall of Constantinople presumably knew of this practice. Putting one in his painting added an exotic touch, possibly even an ecumenical note. Meanwhile, as scholars have noted, there was a mystical tradition which linked the ostrich egg with the womb of the Virgin and with the birth of Christ.45 For Piero, however, it also had a third meaning. As an egg which when seen correctly from below (fig. 7.3) transformed itself into a sphere, it was a symbol of the universe demonstrating the power of perspective not only to represent objects three dimensionally but also to transform them systematically. As such it permitted an observer to re-enact optically a version of Cusa's game of the globe in which God uses geometrical transformation to play with the universe at once spiritually and physically.

5. Luca Pacioli

Piero's works were not published in his lifetime. Manuscript copies of his three treatises entered the library of the Duke of Urbino who, apparently made them available to a Franciscan friar, Luca Pacioli, who had been born in the same small town of San Sepolcro as Piero. Pacioli became extremely interested in the regular and semi-regular bodies. By 1489 he had commissioned various models of these bodies. In 1494, as Daly Davis46 has shown, Pacioli used *Piero's Treatise of the Abacus* as the basis for a section of his *Summa[ry] of Arithmetic, Geometry, Proportion and Proportionality*. Two years later when he arrived as a guest of Duke Sforza at the court of Milan, Pacioli began work on his most famous text *On Divine Proportion* which he finished in 1498 and published in 1509. On the surface it is not original. Pacioli cites Plato's cosmology and Euclid's geometry as a starting point for his discussion of the regular and semi-regular solids. Scholars have discovered that Pacioli also borrowed heavily from Piero's *Booklet on the Five Regular Solids*.47 Hence it has become fashionable to dismiss him as a plagiarist. But this does not do him justice.

As mentioned earlier, Plato in his *Timaeus* described the composition of all five regular solids, but believed that only three could be changed into one another.48 Pacioli believes that all five are interchangeable. So too does Leonardo da Vinci49 Plato's *Timaeus* as it has come down to us, had no illustrations. Euclid's text, as we have already noted, had diagrams which were spatially unconvincing (fig. 2.1-5). Pacioli, by contrast, commissioned a magnificent set of illustrations by Leonardo da Vinci (fig. 8-11, pl.1,3,5). The opening lines of his preface confirm that this was not merely a decorative flourish. Pacioli cites Aristotle to claim that sight is the beginning of wisdom50 and to strengthen his case he uses another of Aristotle's phrases which the mediaeval philosophers had used quite differently: that there is nothing in the intellect which was not previously in the sense.51

It is quite true of course that Aristotle tended to praise sight above the other senses. But neither Aristotle nor any of the ancient philosophers made clear distinctions between sight a) in a mental sense of something in the mind's eye and b) is a physical sense of things seen by the eyes. In Pacioli's interpretation the focus is clearly on the second of these, i.e. on visual information and then in a rather special sense. For as Pacioli presents it, optics and perspective, that is, vision and representation are fully interdependent. Pacioli suggests, moreover, that there are connections between visual demonstration and mathematical certitude, which leads him, in the next chapter, to propose a new version of the seven liberal arts. The mediaeval tradition had favoured three arts (grammar, rhetoric and dialectic) and four sciences (the so-called *quadrivium* of arithmetic, geometry, astronomy and music). Disciplines such as optics, architecture and geography were seen as dependent upon these or classed simply as mechanical sciences. Pacioli pleads that perspective in the sense of both optics and linear perspective should become the fourth science, and that in terms of importance it deserves to be put into third place, directly after arithmetic and geometry.52

Hence while citing Plato, Aristotle, Euclid and other standard authorities, Pacioli emphasizes perspectival demonstration in a way they could not have imagined. His reasons for writing are also very different. First the unity of proportion, and its indivisible nature is a symbol of God. Second the three terms of proportion symbolize the Trinity. Third the irrationals of proportion reflect the mysteries beyond the rational involved in God. Fourth, God's immutability is reflected in the unchanging laws of proportion which apply to quantity be it discrete or continuous, large or small. Like the ancients, Pacioli sees proportion as the basis for construction of the five regular solids and thus as a key to the nature of trhe four elements on earth and the ether of the heavens above53. But whereas Plato stopped at five bodies, Pacioli consciously refers to "infinite other bodies"54 dependent on these.

The actual number that Leonardo illustrates (fig. 8-11) is somewhat less: 40 to be precise plus an appendix with twenty-one variations of columns and pyramids. Nonetheless, the systematic approach that underlies their presentation is striking. Thirty-four of the figures relate to the five regular bodies, arranged in the order pyramid, cube, octahedron, icosahedron and dodecahedron. In each case both a solid and an open version is given, first of the regular body, followed by its truncated form and then its stellated form. In the case of the cube and dodecahedron the stellated versions are truncated in turn. Among the 34 shapes thus produced are five of the semi-regular Archimedeian solids. The next shape is a twenty-six sided figure, technically called a rhombicuboctahedron, which is again one of the Archimedeian solids. Its appearance both here and in Pacioli's portrait (fig. 42.1.) is probably no coincidence. Since ancient times this shape had mystical associations. In the museum at Aquileia, for instance, there is an antique rhombicuboctahedron so constructed that light in the shape of a crescent moon appears at its surface (fig. 1,5). Finally there is, in the *Divine Proportion*, a seventy-two sided figure which Euclid had described in Book XII, proposition 10 of his Elements and which symbolized perfection during the Renaissance.55

If Pacioli's ideas were borrowed no one before him had ever presented them so clearly, systematically or eloquently. What had previously been an obscure philosophical matter now became a topic of interest at court. Copies of the manuscript went to members of the duke's family. Physical models of the solids were made. In 1504 the town council of Florence commissioned Pacioli to make models for them. In August 1508 in Venice, Pacioli even gave a sermon on proportion to leading noblemen and scholars56, which he published the following year as the preface to Book V of Euclid's *Elements*. Polyhedraphilia had begun.

6. Leonardo Da Vinci

The popularity of Pacioli's book was due largely to Leonardo da Vinci's illustrations. Leonardo's preparatory drawings for many of these have survived57 and offer some insight into how he worked. In some cases the sketches are so rough that he is clearly visualizing the object as he goes. In other cases his drawings are so polished that he very probably had a physical model in front of him and he may well have used the same perspectival window that he employed in drawing objects such as the armillary sphere (fig. 25.1).

There was a little more to Leonardo, however, than a person who made pretty pictures following someone else's instructions. We find, for example, that the notebooks contain various other solids not included by Pacioli. Among them are preparatory sketches of the seven other Archimedeian solids (see Appendix II). This means that Leonardo had represented all thirteen of the Archimedeian solids a full century before Kepler, who is frequently given credit for being the first to do so. Leonardo is also the first known to have made ground-plans of the regular solids or nets to use the modern technical term, a practice that was taken up in a *Modena manuscript* of 1509, then published by Dürer, Hirschvogel (fig. 16.1), Cousin (fig. 23) and has remained a standard aspect of regular solids ever since.

Leonardo's deeper contribution lies in changing the whole context of the discussions. He was well aware of traditional links between vision, perspective and geometrical play. He had almost certainly read Alberti's *On Painting* and he cited Alberti's *Mathematical Games* directly. Leonardo worked with Francesco di Giorgio Martini, who used surveying instruments to demonstrate basic principles of perspective. Around 1490 Leonardo began a systematic study of these principles. This led him to discover the inverse size/distance law of perspective which states that if one doubles the distance of an object its size on the picture plane is half; if one trebles the distance, the object's size is one-third and so on. Leonardo recorded his findings in a thirteen page treatise that is now a section of *Manuscript A* at the Institut de France in Paris.58

Leonardo's demonstrations involved a surveyor's rod, a perspectival window and other instruments, with the aid of which the geometrical properties of visual pyramids would be systematically recorded. Intersections of the pyramid demonstrated perspectival effects in the manner of conic sections.59 In Alberti's *Mathematical Games* transformations were purely a matter of geometry. In Leonardo's *Manuscript A* these transformations remained geometrical but related to visual experience, measurement by instruments and perspectival representation. They were no longer mental abstractions. They could be seen, measured, recorded and represented.

Euclid's version had been with two-dimensional lines. Leonardo's version meant that Euclid's propositions could be expressed three dimensionally. The Pythagorean theorem, for instance, was no longer an abstract geometrical idea: it involved perspectival versions of triangular and square boxes. Euclid had catalogued static lines. Leonardo set out to catalogue the dynamic properties of three dimensional shapes: a 3-D version of the geometrical game.60 His plan emerges slowly. Hints of it are found in his earliest notebooks. But he is 53 before he writes his first serious treatise *On the Geometrical Game* 61 in 1505. It has three books, we would say chapters. It is written in mirror script. As far as a modern reader is concerned even the pagination goes backwards. But we need merely glance at a few pages of this text, with its neat paragraphs and numbered illustrations to recognize that Leonardo is working systematically (fig. 12-13). He is concerned with equivalent areas of pyramids, cubes and rectangles which leads him a few pages later to show how one transforms cubes to pyramids and pyramids into dodecahedrons and conversely (fig. 14.1-2). This leads in turn to an amazing list of twenty-eight kinds of geometrical transformations62 the first twelve of which are simple, i.e. where two changes leave another aspect unchanged, while the remaining sixteen are composite in which all aspects change.

This treatise now in the Victoria and Albert Museum in London marks a first step in a much more ambitious programme that dominates the next eleven years of his life. The manuscript that resulted is lost, but there are enough hints in his preparatory notebooks to give us a vivid idea of his activities. For some time Leonardo remains undecided about a title so we find him referring to a book of equations63 in the sense of equivalent shapes, or a book of transmutations64 in the sense of transformation. He continues to refer to a work *On the Geometrical Game* 65 but its contents change with time. As noted above his treatise of 1505 had listed 28 kinds of transformations. In 1515 he describes the geometrical game as a "process of infinite variety of quadratures of surfaces of curved sides."66 About a year later this has evolved into a treatise in its own right of 113 chapters with 33 different methods of squaring the circle67, which he intends to use as an introduction to his work *On the Geometrical Game*. In his own words:

> Having finished giving various means of squaring the circle, i.e. giving quadrates of equal size to those of the circle, and having given rules to proceed to infinity, at present I am beginning the book on the geometrical game and shall once again give the means of infinite regression.68

By this time, however, *On Squaring the Circle* and *On the Geometric Game* have both become part of a magnum opus on the *Elements [of Mechanical Geometry*] with a second volume on the *Elements of Machines*.69 This second volume was not simply an afterthought. It was again something on which he had been working for almost thirty years. Initially, as an engineer he had become struck how machines involved a surprisingly limited number of parts such as gears, screws and rivets. He catalogued 21 of the 22 parts known today. While doing so Leonardo became convinced that these must be governed by more fundamental principles or powers. By 1492 he was convinced that there were four basic powers: weight, force, motion and percussion.70 To explore their properties he made experiments with weights and balances, pulleys and other mechanical devices and discovered that the powers had proportional variations.

As a theologian, Pacioli had been interested in proportion mainly as a stimulus for religious meditation, as a key to understanding God himself. By contrast, Leonardo, as a scientist, was concerned with proportion as a means for understanding God's creation: the natural world. For a time he pursued this goal in terms of two separate research programmes. One focussed on pyramidal proportion and involved perspective, optics, transformational geometry, surveying and painting. A second programme involved proportions in mechanics and physics. As the 1490's progressed he gradually hit upon the idea that pyramidal proportion offered a key to both programmes. As he put it in a note in 1500:

> All the natural powers have to be or should be said to be pyramidal, that is, that they have degrees of continuous proportion towards their diminution as towards their growth. Look at weight, which in every degree of descent, as long as it is not impeded, acquires degrees in continuous geometrical proportion. And force does the same in levers.71

For Leonardo proportion was "not only in numbers and measures but equally in sounds, weights, tones and sites and every power that exists."72 Further experiments convinced him that the pyramidal proportions of perspective involved a pyramidal law that applied to all dynamic situations falling into two basic categories: first, changes in shape as in transformational geometry; second, changes in weight, motion, force and percussion (including optics, acoustics, and heat) in mechanics and physics. By 1516 these two classes had inspired his two volumes on the *Elements [of Mechanical Geometry*] and the *Elements of Machines.* As far as Leonardo was concerned these works were his version of a unified theory.

At the level of practice two other projects had also defined his life's work. As a young man Leonardo had set out to write a basic work on the microcosm which blossomed into his anatomical studies. He also planned a companion work on the macrocosm. Based on his optical and astronomical studies this was intended to offer a new cosmology. He envisaged that his *Elements of Geometry* and *Elements of Machines* could serve as a theoretical foundation for his *Anatomy* and *Cosmology*, but unfortunately he died before he could publish his new encyclopaedic synthesis.

A generation earlier Cusa, building on Plato's Timaeus and Euclid's *Elements* had used proportion and regular solids as a means of understanding God. Pacioli shared Cusa's views of mathematical theology with the exception that where Cusa relied on intellectual images of geometrical transformation, Pacioli insisted on perspectival examples of the regular and semi-regular solids on which these transformations were based. Leonardo drew the images that Pacioli envisaged. He also changed the context in which they were seen. The regular solids, the geometrical game, the whole of Euclidean geometry became part of a new approach to science that was visible, quantitative and reversible. Indeed, geometrical transformation now became synonymous with science itself. As Leonardo put it:

> If a rule divides a whole in parts and another of these parts recomposes such a whole then one and the other rule is valid. If by a certain science one transforms the surface of one figure into another figure, and this same science restores such a surface into its first figure then such a science is valid. The science which restores a figure to the original shape from which it was changed is perfect.73

For Leonardo's predecessors the regular solids were stimuli for religious meditation. For Leonardo they became building blocks of reality, revealing the structure of the universe, accounting not just for static objects, but for all changes of shape therein. The regular solids were no longer merely abstract symbols linked with a world of ideas. They were intimately connected with the physical world, were models of reality74 and as such could be physically represented, reconstructed and measured. Even their transformations could be computed mechanically. This, as we shall see was exactly what happened in the generations following Leonardo.

A first reaction to these innovations was simply to make physical models of the regular solids. In (fig. 43.1) the famous anonymous portrait of Pacioli, for instance, we see a model of a dodecahedron in the lower right. In the upper left there is a glass model of the twenty-six sided figure, or rhombicuboctahedron which Leonardo also illustrated in Pacioli's book *On Divine Proportion* (pl. 1). In both cases the model is suspended. But in the painting it is also half filled with water and as Dalai-Emiliani75 has acutely noted involves unexpected optical effects. If we look at a detail (pl. 2), we see at the upper left of the rhombicuboctahedron a reflection of a window in a Renaissance palace. It is almost certainly Urbino since Pacioli is shown with the young Duke of Urbino looking over his shoulder. This image of a window is reflected a second time on the surface of the water whence it is refracted to the lower right hand surface.

The Duke's interest in the subject apparently extended well beyond looking over Pacioli's shoulder. There is a story still told by the priests of Urbino today that he chose a stellated figure of the dodecahedron (pl. 3) as his personal symbol and had lamps made in this form. There is one in the basement of the cathedral. Thus far I have been unable to find documentary evidence so the story may well be apocryphal. Nonetheless, there is a shop in Urbino, which thrives in selling beautiful reproductions of the so-called ducal stellations.

Others were also fascinated by these forms. Brother Giovanni of Verona, a monk who was among the leading masters of inlaid wood in his day adopted a number of Leonardo's illustrations for his own purposes. For instance, in the choir stalls of Monte Oliveto Maggiore, near Siena, he included both a stellated dodecahedron (pl. 4) and a seventy-two sided figure (pl. 6). Later in the sacristy of Santa Maria in Organo76 in his native city of Verona he again used this seventy-two sided figure, this time a combination with a twenty-sided icosahedron and its truncated form (pl. 7).

If all these solids were taken directly from Leonardo's illustrations in Pacioli's *On Divine Proportion*, their meaning was based on Pacioli himself. These were symbols intended to inspire religious meditation, and this remained the norm in Italy during the first decades of the sixteenth century. It was elsewhere that Leonardo's interests in practical and scientific transformation were first appreciated.

7. Nürnberg Goldsmiths

In early sixteenth century Germany it was mainly the practical aspects of regular solids that aroused interest. Bits of evidence confirm that Leonardo had some influence on Albrecht Dürer in this respect. Several of Leonardo's anatomical drawings recur in Dürer's notebooks. Leonardo's perspectival window (fig. 25.1) recurs in Dürer's *Dresden Sketchbook* in 1513, in the *London Sketchbook* in 1515, and then in his published *Instruction of Measurement* of 1525. More important for our purposes, Leonardo's drawing for a dodecahedron is published by Dürer in the same book. But neither here nor elsewhere is Leonardo acknowledged. Had they actually met, as some scholars have rumoured, then one would have expected Dürer to report on Leonardo's scientific world view.

Indeed in his *Instruction of Measurement*, Dürer cites only Euclid in connection with the regular solids. However, what had been abstract problems of mathematical construction for Euclid, are concrete and practical for Dürer. He describes, for instance, how each of the five solids can be constructed physically, how to construct a sphere within which these solids can be inscribed, fit models of one solid within another and add pyramids to their sides to produce their stellations. His instructions for semi-regular solids are even more vivid:

Much more beautiful bodies can also be constructed which can again be inscribed with all their corners within a hollow sphere, but these have uneven sides. Some of these I shall draw in plan completely, such that anyone can put them together. Whoever wants to make his own should draw them larger on a cardboard of double thickness and cut them with a sharp knife such that one cuts through the one layer through to the next. When one then folds the body together it can readily be bent at the cuttings. Hence pay attention to the following drawings since such things are of manifold use.78

Dürer then describes and illustrates the nets of six Archimedeian solids and three variants (cf. Appendix I) adding that:

> if one uses sharp scissors and cuts away the corners from these examples and then cuts away the remaining corners one can in this way construct a number of other bodies. From these things one can make a good number of objects where one part is placed on top of another, which is useful in sculpting columns and their decorations.79

Where Leonardo was concerned with principles for constructing and representing solids, Dürer provides the equivalent of a how to do it book. For Leonardo the solids were the key to a scientific understanding of the universe. For Dürer they provide practical hints for architectural decoration. To be sure the *Instruction of measurement* is also about many other subjects. It deals with measurement in the context of architecture, instruments, transformational geometry and principles of perspective. In a larger sense it is about mechanical means of construction and representation. It had a seminal influence. It was soon translated into Latin and subsequently into French.80

Some of the work connected with Dürer's workshop remained unpublished. Some drawings illustrated, for example, how the method of truncation, as described by Dürer in the passage just cited, actually worked when applied to a tetrahedron (fig. 15.1). Other drawings confirmed that his workshop was also experimenting with a) how different orientations and shadings could be used to produce different effects in a given shape (fig. 15.2); b) how new shapes could be generated by careful geometric play (fig. 15.3) and c) were using anamorphosis, i.e. deliberately distorted versions of solids such as a truncated cube to create new effects81 (fig. 15.4). This is of considerable interest for our purposes because all these developments are used in the masterpieces of inlaid wood a few decades later (pl. 53-54).

Six years after the first edition of Dürer's *Instruction of Measurement*, an anonymous *Beautiful Useful Booklet* 82 (1531), edited by Rodler set out to popularize Dürer's ideas. But this scarcely mentioned the regular solids. Twelve years passed. Then came 1543, which was an important year for the history of science. Tartaglia published the first vernacular edition of Euclid. Copernicus' *Revolutions of the Heavens* and Vesalius' *Fabric of the Human Body* also appeared that year. More relevant for our purposes, Augustin Hirschvogel published his work on *Geometry with the curious subtitle: The book geometry is my name, all liberal arts at first from me came. Architecture and perspective together I bring*.83 It contained some basic propositions on perspective but was mainly about polyhedra. Besides the regular solids, Hirschvogel considered seven of the thirteen Archimedeian solids (Appendix**). His novelty lay mainly in his presentation. Where Dürer gave only ground plans, Hirschvogel provided them in combination with different views, thus correlating two- and three-dimensional versions of an object (fig. 16.1-3). Even so his emphasis remained on the practical architectural applications of these solids. Lautensack's treatise on perspective (1564), which again contained both regular and semi-regular solids (fig. 17.1-4) continued this tradition.84

This changed with the publication of Jamnitzer's *Perspective of the Regular Solids*. Wenzel Jamnitzer85 (1508-1585) had been born in Vienna and arrived in Nürnberg in 1534. In the decades that followed he became one of Nürnberg's leading citizens86, and Europe's most famous goldsmith. While clearly interested in the practical applications of these forms in making jewelry and ornamental vases, Jamnitzer was also fascinated by the cosmological aspects of the solids as he indicated in his long subtitle:

> that is, a diligent exposition of how the five regular solids of which Plato writes in the *Timaeus* and Euclid in his Elements are artfully brought into perspective using a particularly new, thorough and proper method never before employed. And appended to this a fine introduction how out of the same five bodies one can go on endlessly making many other bodies of various kinds and shapes.87

Jamnitzer never managed to to publish his fine introduction. Nonetheless, his method was clear. Part one had five sections, each headed by one of the five vowels (a, e, i, o. u), corresponding to one of the five regular solids and one of the five basic types of matter: earth, air, fire, water, heavens (pl. 9-13). Each section contained 24 illustrations, i.e. a solid followed by 23 truncations and stellations. Scholars have suggested that Jamnitzer's 24 variations were an allusion to the 24 letters of the Greek alphabet[88], in which case these shapes are a metaphorical alphabet of Nature's forms. Later writers such as Lencker and Halt were explicit about these comparisons. Part two opened with five pages again headed by five vowels, each showing two transparent regular solids mounted on a stand (e.g. pl. 14-15). Six pages of variations on the seventy-two sided figure followed (e.g. pl. 16-17). The first of these was based directly on Leonardo's illustration (pl. 5) which Brother Giovanni had used in Verona (pl. 7). Four pages of pyramids (pl. 18-19) and three pages of cylinders (e.g. pl. 20) completed the book. Jamnitzer's work remained popular, was reprinted and went through three pirate editions in the early seventeenth century.[89]

Hans Lencker, Jamnitzer's younger contemporary, was also a goldsmith. His first work was a *Perspective of Letters* [90] (1567) which represented all the letters of the alphabet perspectivally as if they were semi-regular solids. (pl. 21-22). The idea had evolved ever since Pacioli included letters as an appendix to the solids in his *Divine Proportion.* Albrecht Dürer and Geoffrey Tory[92] also explored both themes together. However, it was Lencker who first represented the whole alphabet three-dimensionally. His first edition (1567) contained only illustrations. The second edition (1595) added a preface noting that these forms were the true elements and first principles which one needed to learn all other disciplines.[93] A generation later Peter Halt (1625) took this reasoning further: just as one could not speak without vowels, one could achieve nothing in terms of perspectival drawing without the regular solids.[94] This spelled out, so to speak, Jamnitzer's earlier use of the five vowels and the five solids. In Lencker's book these perspectival letters took up the first thirteen folios of the book. These were followed by striking examples of semi-regular forms in combination.

Where Jamnitzer had concentrated on making visible the programmatic elements of his method, Lencker concentrated on demonstrating his perspectival prowess. His first figure showed three cylinders learning around a pyramid, each cylinder mounted by a cross on top of which were balanced, in trapeze artist style, a series of seven six-sided stars, the central one of which had poised on it a diamond shaped object (pl. 23). A third showed interesting rings on a stand (cf pl. 30). Another showed an open cube inscribed within an open twelve sided shape balancing on a stand (pl. 24). The final illustration was a stellated shell (pl. 25). Lencker's main work, *Perspective* (1571) contained examples of a truncated cube, an octahedron and a dodecahedron on a stand. In the tradition of Dürer and Lautensack, he was interested in the regular solids mainly in terms of architectural applications. Lencker's originality, as we shall see later, was in inventing new instruments for representing and constructing these regular solids.

Closely related to both Jamnitzer and Lencker is an anonymous manuscript now in the Herzog August Bibliothek in Wolfenbüttel. Many of its 36 regular and semi-regular solids are so clearly derived from Jamnitzer that Franke (1972) has attributed the manuscript to him. Several factors argue against this. Since Jamnitzer and Lencker were rivals it is highly unlikely that Jamnitzer would simply have copied poorly a figure based on Lencker (pl.30) without improving on it. The execution of the figures, while exquisite lacks the sharpness of line characteristic of Jamnitzer (pl. 9-13) or his engraver Jost Amman (pl8). Moreover, as we have seen Jamnitzer had metaphysical concerns which would not have been in keeping with the playfulness of these shapes (pl. 30-31), nor with the entertaining animals that accompanied them (e.g. pl. 27, 32), which are more than a little reminiscent of the side panels of the Münster cabinet (pl. 57-58). Was the anonymous author of the manuscript from Augsburg and possibly from the same workshop that produced the cabinet?

Younger than both Jamnitzer and Lencker was a third individual about whom we know all too little. We know that Lorenz Stoer was a painter and illustrator, that he was active in Nürnberg until 1557 when he moved to Augsburg where he died around 1621. He is credited with one book, *Geometry and Perspective* 95(1567) which involved a series of eleven woodcuts showing combinations of the regular solids in a landscape with ornamental shapes (pl. 33-36). These were intended as designs for inlaid wood. The work was popular enough to go through a second edition. While I have not yet found any furniture, which used precisely these motifs, the remarkable chest now in Münster was clearly inspired by Stoer's ideas (pl. 56-58).

Not generally known is that Stoer produced two other illustrated manuscripts. One of these, without a title page, is a collection of 33 hand painted illustrations now in the Herzog August Bibliothek in Wolfenbüttel96. The first folio contained only two line drawings of a cube, the second a cruciform figure. The remaining 31 folios each contained semi-regular solids. A majority of these were truncations resulting in sphere-like shapes. Some were stellations (pl.37-38). Three were variants on cylindrical forms and two were based on pyramidal shapes. Two were composite with either a series of regular and semi-regular solids (pl.39) or in combination with a series of crosses (pl.41). The other manuscript, a magnificent collection of 336 folios with over 640 solids of such illustrations, is now in the University Library at Munich. This is in six parts, beginning with *The Five Regular Solids Cut in Various Ways* 97 which deals systematically with the five regular solids in various truncations and stellations. Using a related method Jamnitzer had produced 120 solids (pl. 9-13). Stoer's method generated ** solids (e.g. pl. 43). Part two, entitled *Geometrical and Perspectival Bodies* 98 included a series of further truncations and stellations (e.g. pl. 44). A short third section focussed on conic and cylindrical shapes99 (pl. 48). More truncations and stellations followed in a fourth section on *Geometry and Perspective*.100 A fifth section showed these objects piled two, three and fourfold on top of one another101 (pl. 45-47). A final section entitled *Geometrical and Perspectival Regular and Irregular Bodies* 102 contained further dramatic examples (pl. 42, 49-50).

Stoer's manuscript was actually a compilation of over three decades' work ranging from 1562 through 1599. The novelty of his remarkable effects lay mainly in his combination of earlier techniques. We have noted, for example, that Dürer's workshop explored the use of shading to enhance the spatial effects of these solids. Jamnitzer and Lencker developed this technique using narrow banks of colour to accentuate the borders of these shapes. Stoer added a feature of his own. He used the surfaces of his solids as spaces in which to inscribe further polygonal shapes. Frequently he combined both techniques. In the case of a dodecahedron, for instance, he outlined the boundaries of its twelve surfaces with bands of colour. In each of these he then inscribed a pentagon.

The beauty and effectiveness of Stoer's shapes depended in part on their colour, which may well explain why these manuscripts were never published. There was no colour printing at the time. It also helps us to understand why these motifs became popular in furniture. Here narrow bands of colour could readily be adapted as stripes of wood, ivory, ebony, etc., which is almost precisely what an anonymous master craftsman did when he produced an inlaid writing desk top now at the Museum for Decorative Arts in Frankfurt (pl. 58). Indeed, although clearly not a copy of Stoer's illustrations, it is obviously a complex variation of his themes (cf. pl. 57.1). Where Stoer had a central shape topped by an inverted pyramid, the desk has a variant shape, topped by a semi-regular solid and a complex inverted pyramid. The intersecting square shape in the upper left of Stoer's illustration recurs in the upper right of the inlaid panel. The hollow cubic shape in the lower right hand corner of Stoer's illustration--which Leonardo had used two generations earlier, recurs in the upper left of the inlaid desk.

These principles were further developed in a cabinet now in the Museum of Applied Arts in Cologne (pl. 59). At first sight the two central compound scenes surrounded by thirteen drawers of three solids and flanked by at least twenty figures on each of the side panels creates an impression of overwhelming complexity (pl. 60). On looking more closely we find that the two central panels are effectively mirror images of one another, in terms of both shape and colour, i.e. a black strip on one side usually has a corresponding light strip on the other side. Moreover, we find that the two upper innermost figures in the central panels recur as the upper outermost figures in the flanking side panels. Regularity is in fact the underlying theme. The two rows of three solids in the upper central section are identical in shape. In the right hand section of the central panel the upper two rows have a recurrent pattern which is repeated on the left in the second from the top row and the bottom row. Indeed it is likely that the bottom drawer was originally in the top left. In which case both the two upper left and upper right drawers would have been symmetrical, as would all three drawers in the bottom row. A simple switch of the drawers in rows two and three on the left side would make them symmetrical with their counterparts on the right. In other words only five basic patterns underly the thirteen drawers and it is simply through an ingenious play of colour that these five rows of three look like 39 different solids.

The side panels each have two complex scenes flanked by three rows of four semi-regular bodies. The shapes in the left panel are again mirrored by those on the right, with a corresponding alternation of light and dark. The dodecahedron in the complex scene (pl.57.2) is again reminiscent of Stoer's illustration (pl. 43.2) except that where Stoer had only one set of inscribed pentagons this panel has two. In the upper right of the left hand panel is a cross composed of seven cubes. This shape, now called a hypercube was another of those which Leonardo had explored two generations earlier (fig. 18.1). It recurred in Stoer's manuscript, is used four times on these panels (e.g. fig. 18.2) recurred in seventeenth century texts (fig. 18.3-4) and English gardens of the eighteenth century before being rediscovered as a symbol of the fourth dimension in the twentieth century. If one opens the drawers there are further scenes and if one opens these in turn there are even more scenes all in inlaid wood: noble figures, hunting scenes, allegories, scenes of towns, landscapes, animals. A full analysis would require a monograph in itself. For our purposes it is enough to note how these complex plays of semi-regular solids became one layer of a complex Mannerist game of hidden images.

The cabinet is not dated but is likely to have been made between the mid 1560's when Jamnitzer was active in this field and 1585 when there was a plague in Nürnberg. Both Jamnitzer and Lencker were victims of that plague. So too were Jamnitzer's papers. Indeed enthusiasm for the regular and semi-regular solids was never quite the same thereafter. One exception was Paul Pfintzing103. Comparison of manuscripts in Nürnberg (fig. 19.1) and in Bamberg (fig. 19.2-3) allows us to trace how he developed his illustrations. Pfintzing, who was an artist also active as surveyor and map maker, set out to write a summary history of perspective focussed on technical developments in Nürnberg: Dürer, Jamnitzer, Lencker and an otherwise unknown artist Hans Hayden. His imaginative drawings (pl. 52-56) show various perspectival instruments and some of the shapes they were used to produce. Pfintzing planned his book for private distribution to friends. Several printed drafts with manuscript additions now in Wolfenbüttel and Bamberg 104 confirm that he changed his mind several times. The result was a book with highly unusual illustrations105, showing these technical instruments for perspective perched precariously on regular solids.

In the next generation a perspective book by Brunn (1615) contained some regular solids. But the focus of attention shifted to Augsburg and Ulm where there were new editions of Lencker107, Pfintzing108 and Stoer109 in the years 1615 and 1617. There were edited by Stephan Michelspacher who clearly intended a revival of this genre. But in 1618 the thirty years war began. In 1625 there was one last important work by Peter Halt, entitled *The Art of Perspectival Drawing*. Inspired by Jamnitzer's association between the five vowels and the five regular solids. Halt's title page literally showed each of the vowels precariously balanced on its corresponding solid. His woodcuts were frequently original. Yet there was a rough and ready aspect to their execution that was not simply a reflection of difficult times (pl. 69-71). The fascination with these shapes as symbols of Divine perfection remained, but their representation had become less important. Before we can understand this paradox we need to examine developments in Italy and France.

8. Italian Popularizers

In Italy publication of works on perspective proceeded more slowly than one might have expected. Indeed, after Pacioli's Divine Proportion in 1509, there was nothing new until Serlio in the 1540's and as a practicing architect Serlio had little interest in regular solids. The first to pursue these themes was Daniele Barbaro in his *Practice of Perspective* 111 in 1568. Barbaro's book was in nine parts with sections on optics, based mainly on Euclid: planes, solids, architecture, anamorphosis, the planisphere, light and shade, the human body and instruments. Part three, by far the largest section of his book (pp. 43-138), while ostensibly dealing with representation of three dimensional bodies was actually a treatise on the regular and semi-regular solids, ending with a detailed study of the mazzocchio which also served as a leitmotif for his chapter headings (figs. 20.1-6). Barbaro began with an outline of three basic methods of rendering objects perspectivally, then provided plans, elevations and three dimensional versions of the five regular solids, descriptions of nine Archimedeian and ten other semi-regular solids, plus ten stellations three of which he illustrated (Appendix III).

Barbaro's presentation differed from his predecessors. Piero della Francesca, for instance, in the tradition of Euclid, had focussed on a mathematical description of how the solids could be constructed. His diagrams functioned almost as an afterthought to the text. In Pacioli's *Divine Proportion* mathematical description remained, but as we have seen, Leonardo's diagrams vied with the text in importance. Dürer, by contrast, had assumed no mathematical knowledge, and emphasized a hands on approach, describing how each body could be constructed physically. Barbaro's approach was somewhere in between. He implied that the bodies could be physically constructed, but presented them in so abstract a way that a mathematical approach was also implied although not officially required. In some cases, as with the twenty-six sided figure112, he provided a series of viewpoints of a single object (fig. 21.1-5, each assuming an optical approach, yet lacking the spatial presence of Leonardo's version (cf. pl. 1). At other times Barbaro provided only a ground plan which did not always correspond to his description. In his text, for instance, he described an irregular object with 36 squares, 24 hexagons, 6 octagons and 8 twelve sided figures. However, his accompanying diagram showed only 8 squares, 7 hexagons, 3 octagons and 2 twelve sided figures.113 In his text Barbaro emphasized the value of short cut methods and since he no doubt assumed that most readers would use these, he did not worry about his diagrams being accurate.

Any suspicions that Barbaro was not capable of a painstaking mathematical approach quickly disappears if we turn to the manuscripts on which the printed text is based.114 One manuscript contains an unpublished section of over 100 pages dealing with three semi-regular solids in what looks in retrospect as a fanatical attention to detail.115 Indeed we discover that Barbaro had two quite different approaches to illustrations: one was simply to give a general impression of the principles involved, the second was a step by step method using instruments to arrive at a mathematically based visual image.

A generation later this second approach gained a brief supremacy but with two paradoxical results. As the means for technical reproduction of perspectival images were perfected emphasis shifted to principles of reproduction and abstract mathematical figures.116 Egnatio Danti's commentary to Vignola's *Two Rules of Practical Perspective117* (1583) is a good example. Danti dwelt at some length on the principles for producing regular polygons.118 He specifically defined the perspectivist's task as the transformation of geometrical bodies119 and yet he gave no three dimensional examples. To be sure the regular solids did not disappear entirely. A second result of these developments was a focus on stock examples, as in Sirigatti's treatise120 of 1596, which was dominated by variations on the sphere and mazzocchio (pl. 64-68). Some attention to combinations of these solids in producing architectural effects continued in Sirigatti. An unpublished manuscript by Vasari, Jr., the son of the man who wrote the famous *Lives of the Artists*, contained even more striking examples of this121: rows of cylinders and mazzocchio figures, stairs, architectural vaults and columns, multifaceted spheres, regular solids, stellations, and combinations in architectural settings (e.g. fig. 22-23), including semi-regular shapes such as crosses, musical instruments, chairs and barrels. Most of these shapes were not new: they were popularizations of shapes invented by Nürnburg goldsmiths or even earlier by Leonardo.

The supremacy of the mathematical approach soon had further consequences. Even in books on perspective abstract diagrams triumphed over three-dimensional illustrations. Accolti's *Deception of the Eye* 122 (1625), for instance, dealt only with the five regular solids, the sphere and the twenty-six sided figure, and while due reference was still given to model making123, the emphasis was on short cuts in arriving at geometrical outlines124, and references to Euclid's *Elements*. By the 1670's the situation in Italy was much the same as in Nürnberg. Fascination with representing the multiplicity of semi-regular shapes had all but disappeared. What had been a realm of theology and art was now increasingly a branch of abstract mathematics. This was partly due to developments in France.

9. French Mathematicians

In France, as elsewhere, the interpretations of Euclid were many. One was theological and mystical. Two years after Pacioli's *Divine Proportion* (1509), Charles de Bovelles125 published a treatise *On Geometrical bodies* (1511). He was influenced by Nicholas of Cusa's approach whereby geometrical symbolism provided mathematical guidance to the Divine. He probably knew of Pacioli's work. But where Pacioli had been content to make passing analogies between proportion and the Trinity, Bovelles set out to show systematically that the geometry of the polyhedra provided a symbolic demonstration of the power of one in three and three in one, and as such offered a mathematical guide by means of which one could contemplate the mystery of the Trinity.

A second interpretation was practical. This grew naturally out of a mediaeval tradition whereby geometry was treated literally as measurement of the earth. We have already noted that this occurred all over Europe. What set the French examples apart, however, was their emphasis on the regular solids. In 1544, for instance, Oronce Finé, published a treatise *On Practical Geometry* or on *The Practice of Lengths, Planes and Solids, that is of Lines, Surfaces and Solids in Quantities and Other Mechanical Operations as a Corollary to the Demonstrations of Euclid's Elements*.126 In this work Finé, included a chapter to show that polygons and multilateral figures could be measured127 and another chapter on the measurement of the rest of the regular bodies.128 That same year Finé, published a series of small treatises on quadrature of the circle, measurement of the circle, all the polygons and on the planisphere.129 This contained a booklet *On the Absolute Description of All Straight Lines and Multiangular Figures (which are Called Regular) Both Inside and Outside a Given Circle and on a Flat Line*.130 He pursued this theme in 1556 with a *Corollary on the description of the regular solids131 in a given circle using isoceles triangles*, followed by a section on volumetric measurement of cubes, rectangles, pyramids and the like. Regular polygons in two dimensions and regular solids in three dimensions were becoming a key for abstract planimetric and stereometric measurement.

Even so links with practical problems continued. Cousin, for instance, illustrated models of the five regular solids on the title page of his *Book of Perspective* 132 (1560), and described in detail their three-dimensional construction with the aid of compasses. The royal mathematician, Claude de Boissière in *The Art of Arithmetic Containing all Dimension both for the Military Art and Other Calculations*133 (1561) also dealt with the regular solids in terms of construction, measurement and transformation of one form into another using shapes (fig. 14.3-4) more than slightly reminiscent of Leonardo da Vinci (fig. 14.1-2), who had been his predecessor at the court of the King of France. Boissière's treatment of regular solids was followed by a general rule on the quantity of all barrels.134 Here, once again, measurement of ideal solids served as a basis for measuring less regular shapes in the everyday world. Moreover, as the editor of this work, Lucas Tremblay, pointed out, all this was part of a larger plan to provide a synopsis of arithmetic and geometry with a view to uniting discrete and continuous quantities.135 The Greeks, it will be recalled, had dealt with the discrete quantity of arithmetical numbers separately from the continuous quantity of geometrical lines. By the latter sixteenth century the challenge loomed of dealing with both types of quantity together. Both Fermat and Descartes found a solution in the 1630's in what we now call analytic geometry.

10. The Jesuits

The construction and representation of the solids as well as their religious connotations continued to hold a certain fascination. We have seen how Piero della Francesca's egg in the Brera Altar used anamorphosis for both religious and mystical connotations. Holbein further explored these possibilities with the anamorphic skull in his Ambassadors, which became such a famous motif that it was included among the instruments of the French Academy of Sciences.136 Even so it was not until the 1630's that the didactic potentials of anamorphosis were considered seriously. In 1638, a young Franciscan friar, Jean François Nicéron, produced (fig. 24.1-3) a treatise entitled *Curious Perspective*.144 He was 25 at the time. This served as the basis for an expanded version named Optical Magic137 (1642). Almost immediately some of his ideas were adapted by the Jesuit Father, Jean Dubreuil in his *Practical Perspective* 141, a massive three volume compendium (1642-1649). Dubreuil also dealt with the regular solids as if they were models, thus reviving sixteenth century traditions. It was a popular work and a controversial one. It plagiarized ideas from Desargues, getting some of them wrong in the process. It was attacked by the experts, and needless to say it was a great success. It was translated into Latin, German, Dutch and English going through more than 20 editions. Significantly, however, the translations were often abridged versions in which treatment of the regular solids was all but omitted. The tradition did not die out entirely. Subsequent treatises very often included the five regular solids and some other examples. Nilson, in his *Introduction to Linear Perspective* (1812), included more. One of these, reproduced on our frontispiece, was a stellated form framed by a hemispherical niche. The majority of the others were in a garden setting and included regular solids stacked on one another, semi-regular shapes functioning as sundials, a stellated form on a pedestal of a pyramid resting on a cube (pl. 72), a truncated cubic form (pl. 73) based on Jamnitzer (cf.pl. 11.2 top left) again resting on a cube and a more complex stack of semi-regular shapes (pl. 74) again based on a Nürnberg precedent (cf. pl. 32). Nilson, however, was an exception, indeed perhaps a last example of serious interest in the theme. The sixteenth century passion for these shapes had gone. Why this happened is the subject of our next chapter.

II CRYSTALLOGRAPHY, MATHEMATICS AND ART

1. Introduction 2. Instruments 3. Models of the Universe 4. Nature's Models 5. Abstract Mathematics 6. Art 7.Visual Mathematics.

1. Introduction

In chapter one we explored how cosmological, philosophical, mystical and religious aspects of the regular solids linked with a type of metaphysical mathematics to produce a geometrical game based on transformations of shapes. This game had neo-Platonic tendencies, in the sense of being linked with the world of ideas more than the physical world. As such it could have become the ultimate mind game of the Renaissance and the three-dimensional illustrations by artists would have represented mere bravados of abstraction.

This did not occur because as we have already mentioned there was also a practical side to these geometrical games: the challenge of actually measuring their volume and determining how a given volume was affected by changes in size and shapes. Already in Antiquity Archimedes had explored such problems in his work *On the Sphere and the Cylinder*.1 Hero of Alexander2 had pursued them. Subsequently they had been taken up by Leonardo of Pisa3 and had become part of the abacus tradition, which was partly why the regular solids played so prominent a role in Piero della Francesca's *Book of the Abacus*. He too was concerned with measuring their volume. Pacioli's *Divine Proportion* also stood within this practical tradition. Only 34 of his 61 illustrations dealt with the five regular solids (figs. 8-11). Four illustrations involved a 26 sided shape, two illustrations involved a 72 sided shape, both of which he explicitly discussed in terms of their practical applications to architecture.4 The final 21 shapes were all variants of columns and cylinders. Here again Pacioli emphasized their practical applications for architecture adding a chapter in which he outlined how they could be measured,5 citing Archimedes' work on *Quadrature of the Circle*6 and alluding to a treatise on measurement of the regular solids7 which he had dedicated to Guido Ubaldo, the Duke of Urbino. The sixteenth century continued this practical tradition and also transformed it by developing a series of instruments for both the representation and measurement of these solids.

2. Instruments

The simplest of these was the perspectival window which was probably invented by Brunelleschi in the early fifteenth century. Leonardo's drawing of a perspectival window in recording an armillary sphere is the first extant example of this device (fig. 25.1). Dürer published a version of this window in his *Instruction of Measurement* (1525). So too did Rodler (1531) in the popular version that he edited. Thereafter it became a stock image in perspective treatises. In Nürnberg, Dürer also developed variants of the perspective window. These were improved upon by both Jamnitzer and Lencker and were subsequently published by Pfintzing (fig. 19). For our purposes they are of interest because they illustrate how perspectival instruments led to a new type of measured drawing. This is all the more significant because both Jamnitzer and Lencker were also involved in the development of universal measuring devices. Jamnitzer, for instance produced a special instrument for systematic measurement of various metals, accompanying which he wrote a treatise dedicated Prince August of Saxony in 1565. In 1585 he wrote a more comprehensive treatise on various instruments and surveying practices now in the Victoria and Albert Museum. Lencker also developed his own instruments, which he illustrated and described in his *Perspective* (1567). Measured representation and systematic volumetric measurement thus went hand in hand.

The development of these universal instruments included the measurement of two-dimensional polygons and three dimensional solids. These methods evolved gradually. In the case of polygons Fin,8 had outlined several systematic approaches in his booklet of 1544 without instruments. Implicit in his approach was the principle that different polygons subtended different angles within a circle.9 Danti10 in his commentary to Vignola's *Two Rules of Practical Perspective* (1583) described how this principle could be applied to a circular surveying instrument, an idea that Coignet11 and others carried out in practice. The brothers Fabrizio and Gaspare Mordente found another solution. One could record the diameter of a circle as a line and then mark off the relative lengths of a triangle, square, pentagon, hexagon, etc. inscribed within this circle.12 According to their own account13 they developed this while on a voyage to India in 1554. It was written up in 1578 and published in 1584. A third approach was to record this information on a sector specifically designed for this purpose. This idea is usually associated with Guidobaldo del Monte14 in Urbino around 1569, although the idea of using lines on a sector dated back to 1509.15 The first published version of the Guidobaldo type sector was by Gallucci16 in 1594 (cf. fig. 26.1-2).

Ever since Antiquity practical concerns had prompted three-dimensional volumetric measurement. From the thirteenth century onwards the wine trade provided a special stimulus involving the measurement of wine barrels and led to an independent body of literature on gauging.17 Some cities such as Nürnberg and Antwerp even had a gauging master. By the sixteenth century gauging rods specifically designed to measure the volume of barrels had been developed. These rods usually had lines involving square roots or cube roots. Problems of volumetric measurement also arose in the military with cannonballs of varying sizes and different metals, which led to the development of calipers for these purposes. By the 1550's there were efforts to find a single instrument which would solve all problems of measuring lines, surfaces and volumes.18 Mordente's compass and ruler were an early attempt. Besides dealing with polygons, they dealt with volumetric measurement of pyramids, cubes, spheres and with the transformation of spheres into cubes.

The reduction compass was also used in these efforts (fig. 26.3). It had originally been invented in Antiquity.19 Around 1500 it was developed by Leonardo da Vinci who referred to it specifically as a proportional compass.20 In the period 1560-1580 Wilhelm IV further developed this instrument in terms of seven operations, the last of which is of particular interest for our purposes:

1. To divide a given straight line with a given proportion.
2. To divide a given circular line into various parts.
3. To multiply or diminish a given surface into a surface of the same shape.
4. To multiply or diminish a given body into a body of the same shape.
5. To find the ratio of any diameter to its circumference.
6. To transform some circular or square surface into another one.
7. To transform a given sphere and the five regular solids into one another.21

Wilhelm IV was the Landgraf of Hesse and passionately interested in science. In 1561 he started the world's first modern astronomical observatory at Kassel. Both Tycho Brahe and Kepler had connections with his court.22 In 1579 Jobst Bürgi joined the court as an instrument maker. He developed a reduction compass which carried out the operations listed above.23

The seven operations in Wilhelm IV's instrument are the more intriguing because they recur in a manuscript entitled Perspective attributed to Hans Lencker mentioned earlier.24 Although it has the same title as his treatise of 1571 the manuscript contains 47 additional pages of handwritten text and illustrations. Lencker was based in Nürnberg, but he also travelled. From 1572 through 1576 he was mainly in Dresden where he taught Prince Christian I at the court of Saxony. In 1574 Lencker also had commissions for the courts of Munich and Kassel. It is likely that he would have learned about the Landgraf's manuscript at that time.

The manuscript attributed to Lencker was not simply a copy of the Kassel manuscript. It listed the same seven operations but then discussed them in connection with both a reduction compass (fig. 26.3) and a sector (fig.27.1). The text was more detailed. Some of the diagrams such as those relating to comparative weights of metals were new. Others relating to volumes of spheres showed principles familiar from the gauging literature. A number of the diagrams were clearly based on the Kassel manuscript including the illustrations of the regular solids. In 1604 Levinus Hulsius25 published a report of Bürgi's reduction compass which borrowed diagrams from the manuscript ascribed to Lencker. In 1605 Horcher published the principles underlying this compass.26 In 1606 Galileo published his own version of the sector27 and claimed precedence for the invention. So too did others. This led to a lawsuit. Galileo won. But the debates continued. Neither the details of these debates nor the contents of the 120 books published on the subject and many related instruments (cf. fig. 27.2) in the century that followed need concern us here.28

Important for our purposes is how these new instruments effectively mechanized the basic processes of the geometric game: two-dimensional quadrature of the circle, three-dimensional cubature of the sphere, problems of doubling the volume of a cube or transforming one regular solid into another were now operations which could be analyzed quantitatively. They were physical, mechanical problems which could be reduced to numerical ratios and these could happen without the aid of three-dimensional representations. Hence it was paradoxically the very study of the regular and irregular solids as concrete physical models that brought about a new level of abstraction, which resembled the earlier neo-Platonic interpretation but was in fact fundamentally different because it assumed a new mechanistic view of the universe. Indeed, where the geometrical game had been an intellectual play of geometrical forms in mediaeval times, it now involved nature itself.

3. Models of the Universe

If this change occurred slowly, it is fascinating to note that it involved precisely the individuals whom we have been studying, notably, Pacioli, Leonardo, Jamnitzer, Lencker, and Kepler. Pacioli was fully aware of Aristotle's objections to Plato and was probably aware of mediaeval debates on the subject. But whereas his predecessors saw the regular solids as the source of contention, Pacioli interpreted them as a solution to the debate.29 For him the fact that the regular solids could be snugly fitted inside one another presumably resolved the problem of a potential vacuum and it may well be that he began building models in 1489 partly by way of demonstration. Two decades later when Leonardo da Vinci[30] challenged Plato's ideas in the *Timaeus*, he did so on the basis of experiments with actual solids. He had discovered that pyramids (tetragons) were more difficult to roll than cubes (hexagons): i.e. they were more stable. For this reason he claimed that the pyramid should symbolize the most stable element earth, while the cube should symbolize fire, thus reversing Plato's order.

While Dürer was very much concerned with physical models of the regular solids he did not discuss how this related to their symbolic nature. Jamnitzer, by contrast, is said to have deliberately improved Dürer's perspectival instrument in order that he could represent the regular and irregular solids more accurately. As we have seen Jamnitzer specifically associated the solids with the elements and the heavens, quoting Plato, but meaning something very different. For whereas Plato was referring to something in the world of ideas, Jamnitzer was concerned with something physical and his three-dimensional record of these physical models was his way of getting at their truth. The models were no longer symbols or even models in the sense of replicas. They corresponded somehow to reality itself.

Indeed it is difficult to imagine what other incentive could have prompted him to go to such pains. For we are told that both Jamnitzer and Lencker, after they had made their drawings, carefully coloured them and frequently arranged them in striated panels (*tabulas striatas*).[31] Scholars have interpreted this to mean that they employed them for anamorphic effects.[32] Of this there is no evidence. But there is a more obvious interpretation, namely, that the striated panels were panels of inlaid wood. In this context the close parallels noted earlier between Stoer's manuscript and the inlaid wood of the desk at Frankfurt or the cabinet in Cologne become the more significant. For if the manuscript has painted strips, the furniture literally has striated panels (*tabulas striatas*).

If we look more closely at Stoer's painted examples (e.g. pl. 51.1) we discover that the colours are not just ornamental. They enable us to see and distinguish how various solids are nested within one another. Contemporary sources report that Jamnitzer and Lencker developed these techniques and that the painted interiors which resulted were so masterfully arranged in their precision and foreshortenings that the sight of them caused hallucinations.[33] Whether modern readers will see them as psychedelic pictures is not our concern. What interests us here is why artists at the time made these tremendous efforts, which becomes understandable when we recognize that these solids were intended to represent the elements of the cosmos, models of the universe which they believed corresponded to reality itself. Hence the resemblance between these three-dimensional drawings, Kepler's model of the universe in terms of the five regular solids nested within one another and physical models of the spheres (fig. 28.1-3) was no coincidence. In the minds of the Nürnberg artists these were lessons in cosmology. Indeed, in one of Stoer's examples we can clearly see a dodecahedron (the heavens) within which is nested an icosahedron (water) and other regular solids (pl. 51.1).

In this context the term striate panels (*tabulas striatas*) takes on new meaning. For we find that the term *stria* [34] recurs in Kepler's description of nature, although the *Oxford Dictionary*, which defines *striate* as "marked or scored with striae, showing narrow structural bands, striped, streaked, furrowed," claims that "the earliest examples relate to the hypothesis of Descartes as to the striate or channeled condition of the constituent particles in nature."35 Were these geometrical games of the Nürnberg artists sources for Kepler's and Descartes' geometrical views of nature? We know that pirate editions of Jamnitzer appeared in 1608, 1618 and 1626 in Amsterdam where Descartes lived. Did Jamnitzer's work then have a direct effect on Descartes?

4. Nature's Models

In any case the examples of Jamnitzer, Lencker and Stoer clearly document an important trend. The regular solids were no longer simply symbols of a reality that existed only in the world of ideas. Constructing these solids, representing them, measuring them was a means of studying nature. Implicitly the solids were nature. The 24 year old Kepler pursued these ideas in his *Cosmological Mystery* 31 (1595) where he made a systematic study of both the regular Platonic and thirteen semi-regular Archimedeian solids (fig. 29.1-2). Like Plato, he linked the regular solids with the elements and the heavens. But whereas Plato had dealt with these problems in terms of two-dimensional polygons, Kepler used three-dimensional volumetric solids. His aim was to relate the geometric ratios of the spheres to distances of the planets from the sun as computed by Copernicus. Kepler studied the ratios of number of sides, edges and corners of the solids and the ratio of circumscribing and inscribed spheres, building directly in the computations of Foix and Clavius. This led him to relate the ratio of inscribed to circumscribing spheres, with ratios of inner to outer planetary orbits.

Kepler was aware of a magnetic like force emanating from the sun, which determined the planet's paths. In 1602, he related the area of this spreading force to the velocities of the planets. In his more down to earth activities Kepler was very much concerned with the measurement of wine barrels. Here his careful measurements transformed what had been a rough and ready approach into a proper science of volumetric measurement. In his early studies of the heavens Kepler had compared numerical values for periods of the planets with the areas, velocities and radii of their inscribed and circumscribing spheres. When he now took into account their volumes he discovered that the volumes of the spheres were proportional to their radii cubed and that their periods squared were also equal to their radii cubed. This became his famous third or harmonic law which he expanded in his *Harmonies of the world* 37 (1619). His revised model abandoned traditional assumptions of circular heavens to accommodate the realities of the planets' elliptical orbits. Observational evidence triumphed over tradition and theory. Yet the solids remained. In his new model there were the five Platonic solids plus a sixth regular solid, the small stellate dodecahedron.

Meanwhile, the terrestrial role of the regular solids had gained in significance through Kepler's booklet on the *Snowflake* 38 (1611). In this work he drew attention to the hexagonal pattern of snowflakes, noting how this form recurred in beehives and in the pips of pomegranates.37 He also noted how pentagonal shapes occurred in many botanical forms. In this context he posited the existence of a shape forming power40 and related these basic shapes in nature to the regular solids, particularly the dodecahedron and icosahedron,41 both of which involved the golden section, i.e. precisely that divine proportion which Pacioli had made the title of his book. But whereas Pacioli was interested in the mathematical properties of this proportion for its potential religious symbolism, Kepler focussed on the golden section as a key to understanding the forms of nature itself. He asked himself whether there be a specific purpose in creating snowflakes that are six sided, deciding that the forming power did not only operate with a useful goal in mind, but also with a view to beauty.42 At this point it is worth quoting Kepler directly for he adds that the shape forming power:

> does not only tend to produce natural bodies, but also amuses itself in relaxed games, which is also evident in numerous examples in minerals. The reason for all this I attribute to play (we say that nature plays) with a serious intention.43

In mediaeval thought, God had amused himself with abstract geometrical games. In Kepler's view the geometrical game involves physical objects. God amuses himself by creating unending variations of the hexagonal form in snowflakes44 (Canadians have long suspected that snowstorms are God's secret parties). Or he creates regular geometrical and other fanciful shapes in nature.45 Kepler noted how rock crystals were always hexagonal, while diamonds were in rare cases octahedral.46 For this reason he claimed that the forming faculty "does not limit itself to a single shape. It knows the whole of geometry and is exercised here."47 He reported having seen silver lined copper at Dresden shaped in the form of a dodecahedron48 and cited a description49 of the baths at Bollen which mentioned the front part of an icosahedron amongst the minerals. This led him to suggest that the shape forming power probably differed in accordance with the diversity of humours.50 Rather than embarking on a new theory of chemistry, however, Kepler consciously ended his booklet at this point.

In the Platonic tradition where a world of ideas was paramount there was theoretically no need to deal with its basic forms in anything but mental terms. In practice, however, the Platonic tradition never developed a clear distinction between mental and physical images, between subject and object. Christianity, as noted earlier, introduced a belief that the natural world created by God was real. Implicitly this meant that in order to understand reality one needed to study the natural world. Kepler took this approach to its logical conclusion. He accepted that if the building blocks of nature are regular solids, these too must be physical. Once it was clear that the models were not symbols of a world of ideas but representations of the physical world, attention could shift from deductive models to an inductive programme of searching for all possible examples. Hence scholars have rightly emphasized the importance of Kepler's *Snowflake* for the early history of mineralogy, linking him with the subsequent work of Pieresc, Gassendi, Descartes, and Bartolinus.51 The geometric game had taken a new form: it was now a challenge of trying to find geometric patterns that God had hidden in Nature, a problem of searching for concrete examples (cf. fig. 30-31). Or rather this was one dimension of the story.

Kepler made another contribution which further expanded the horizons of the geometrical game. In ancient mathematics discrete quantity (arithmetic) and continuous quantity (geometry) were dealt with separately although there was a tradition of figured polygonal numbers which allowed numbers to be represented geometrically. There were two methods.52 One used by Euclid represented numbers as straight lines proportional in length to the numbers involved. The other represented numbers by dots or alphas for the units disposed along straight lines to form geometrical patterns. The Pythagoreans developed this concept of number which although represented in physical form was intended metaphysically. Kepler adopted this approach (fig. 29.3) but took literally the concept of a physical form for number, linking this with the external solid form of crystals.53 Planar arrangements of close packed spheres were now both mathematical and physical. So Kepler's new links between discrete and continuous quantity led to further bridges between the concrete physical world of nature and the abstract world of mathematics. Descartes took this further in his Principles where he described such minute spheres in various configurations and noted that he:

> did not accept any principles of physics which were not also accepted in mathematics in order to be able to prove by demonstration all that I will deduce from them and that these principles are sufficient so that all phenomena of nature can be explained by means thereof.54

Through Bartolinus, Hooke and Huygens these minute spherical particles developed into the corpuscular hypothesis55 which led to a new form of atomism.

Not everyone accepted this idea of atomism and an internal molecular structure. The Danish scientist Steno, for instance, concentrated instead on surface form. He adopted Dürer's method for the construction of solids within solids in making his own models of crystals of hematite from the island of Elba. Where others had proceeded deductively in imposing the shapes of the regular solids on nature, Steno proceeded inductively, recording what regular shapes could be found in Nature. As his drawings of quartz crystals reveal, he also recognized that the angle between the faces of crystals was independent from accidents of sizes and truncations.56 By the early eighteenth century Guglielmini (1707)57 and Cappeller (1711)58 succeeded in relating geometric figures of minerals to a limited number of single forms. This approach was developed by Werner (1774)59, who claimed that all crystals derived from specific geometric forms such as the cube, dodecahedron and prism. The same decade saw a series of important advances by Rom, de Lisle (1771),60 Bergman (1773)61 and Buffon (1779)62 which led to Haüy's fundamental work (1801)63 in which he formulated the law relating measured angles of crystalline forms to internal repetition of identical molecules (fig. 32.1-3). Haüy's breakthrough came in the same generation that Berthollet transformed dyeing from a craft to a science and Lavoisier established a scientific basis for chemistry.64 By this time the framework of science was sufficiently well developed that it was no longer a matter of casually adding new facts at random. A programme of finding missing bits of the framework was beginning.

If this prepared the way for modern crystallography it required another century for the next important steps to be made. In 1895 Röntgen invented X-rays. In 1902 his student, Max von Laue explored how X-rays passing through crystals might furnish regular reflections.65 This approach was developed by William Henry Bragg and his son William Lawrence Bragg66, who attempted to show where atoms were located in any crystal framework. Since then crystallography has become a highly systematic science using a variety of methods (e.g. fig. 33.1) to analyse symmetrical properties of crystals. Introductory textbooks67 use stereographic projections (fig. 33.2-3) which are a development of techniques used earlier in the construction of astrolabes and maps. Crystals are classified using six basic systems68, in terms of 17 different ways of regularly arranging points in space69 and divided into 32 basic crystal classes.70

Meanwhile, the international tables for crystallography71 quickly give an impression as to how complex the analysis has become, every mineral now being defined in terms of symmetrical coordinates (fig. 33.3) such that the inductive search for nature's regularities has led to an enormous catalogue of not so regular solids.

5. Abstract Mathematics

As we have seen the regular solids had been a mathematical topic since at least the fourth century B.C. But the scope thereof had been limited largely to constructing these solids in terms of square roots and calculating how they could circumscribe and be inscribed within one another. As noted above the sixteenth century opened the way for change when Oronce Finé wrote an independent booklet specifically on the measurement of polygons. As noted earlier the mediaeval tradition had attributed to Euclid both Book XIV of the *Elements* written by Hypsicles and Book XV written by Isidorus and his pupil. This interpretation continued into the Renaissance and in 1556, the same year that Fin, published his booklet, the Count of Candalle, François Foix, added a sixteenth book to his edition of Euclid.[72] This book was adapted by Clavius[73], the head of the Jesuits, in his important edition of Euclid (1574) which went through half a dozen editions.[74] In the seventeenth century this sixteenth book reappeared in Latin[75] (1645) and was also translated into Dutch[76] (1695) and English[77] (1660) by Newton's colleague, Isaac Barrow. Meanwhile, urged on by King Charles IX, François Foix had added a seventeenth and eighteenth book to his edition of Euclid's Elements[78] (1578) in which he pursued problems of inscribing regular and semi-regular solids within one another. The details of these complex operations need not concern us here. Rather it is Foix's approach that is fascinating. In marked contrast with Pacioli or Bovelles, where religious and mystical considerations were an important part of the discussions, Foix presented his material strictly in mathematical terms (cf. Appendix IV). The stage was thus set for a purely abstract treatment of the solids.

Meanwhile, mathematicians such as Stifel (1543)[79] and Bombelli (c. 1551)[80] began to treat the solids as an algebraic problem. Adrianus Romanus (1593) took this approach considerably further in his *Idea of the First Part of Mathematics or the Method of Polygons in which are Contained the Most Exact and Certain Means of Investigation of the Sides, Circumferences and Areas of any Circle along with Quadratures of the Circle*.[81] Kepler treated both two dimensional polygons and three dimensional polyhedra in algebraic terms.

Paradoxically, however, the same individuals who created new bridges linking concrete nature and abstract mathematics also introduced concepts that threatened to hold them apart. Kepler, for instance, recognized that in order to make the physical a proper object of study required a clear distinction between physical images which can be measured and mental images as in dreams, which cannot be measured.[82] This was a major development in the subject-object distinction and heralded Descartes famous mind-body split. How this dichotomy affected the philosophy of mind has been the subject of much debate.[83] What interests us here, however, is how Descartes' distinction created a parallel dichotomy in terms of approaches to knowledge. Where model making had served as a go between linking nature and mathematics in the sixteenth century, model making now went in two seemingly opposed directions. One, as we noted, linked models with physical objects and led to crystallography. The other linked models with mathematical constructs. On the mathematical side, there emerged, moreover, a tendency to deal with polygons and polyhedra strictly algebraically without any diagrams. While Kepler's work still provided perspectival drawings of the solids, Descartes' treatise on the subject[84] gave no illustrations and relied instead on arithmetical and algebraic charts (Appendix V). By the latter eighteenth century this approach had led to Euler's famous theorem (1758) that the sum of the number of corners and surfaces of a polyhedron exceeds the sum of its sides by two.85 It was in France that the next fundamental advances were made through three individuals: Poinsot (1810),86 Cauchy (1811)87 and Lhuilier (1812-1813).88 As in the case of dyeing, crystallography and chemistry mentioned above, the new system involved more than simply allowing for new examples found at random: it created a framework that had a predictive dimension and provoked deliberate searching for new shapes.

While abstract formulae gained in significance, model-making remained an important activity among mathematicians. An individuals such as Plücker,89 who wanted physical models to explain even algebraic equations, was an extreme example. More influential was Klein, famous for his development of the so-called Erlangen school which emphasized the role of geometry in mathematics and the importance of visual methods generally.90 Klein also lectured on icosahedra and the solution of equations of the fifth degree.91 Günther (1980) who wrote what remains a standard analysis and history of regular and irregular solids used both line drawings (fig. 34.1-6) and photographs (fig.35.1-6) of over 134 physical models.92 Klein's ideas were, in turn taken further by Hilbert93 at Göttingen who developed a visual geometry which made conscious use of both physical models and three-dimensional diagrams in explaining mathematical principles.

Meanwhile the trend towards abstraction in mathematics continued and appeared to triumph. Already at the turn of the nineteenth century when Lagrange published his classic *Analytical Mechanics* (1801) he proudly announced that he had managed to complete his work without using a single diagram.94 This approach affected most branches of mathematics. Indeed algebra, which had originally been linked with practical arithmetic and geometry became increasingly abstract. By 1875 when L”we95 wrote his important dissertation of regular and Poinsot solids and computation of their contents his four diagrams played a marginal role compared to the abstract algebraic formulae that dominated most of the 28 pages of "text." With the work of mathematicians such as von Staudt (1847)96 and van der Waerden (1949)97 a non-visual algebraic geometry emerged as the general category under which descriptive geometry, stereometry and perspective were subsumed. As a result our chief means of visualizing objects and space are classed as branches of a non-visual mathematics. The growing fascination with perspective, regular solids, and crystals in the past decades may well be because we intuitively sense that abstract formulae alone are not enough. Part of this awareness has also come through the efforts of artists and mathematicians who have sought to keep in focus the importance of model making and visual evidence

5. Artists

In 1884 Abbott wrote an imaginative novel in which he compared one, two and three dimensional space and introduced the idea of a fourth dimension.98 Precisely what the fourth dimension might be soon inspired considerable controversy. Some, for instance, thought that time was the fourth dimension. Others believed that it could most effectively be visualized by means of a hypercube.99 Those who did so usually treated it as something new, apparently unaware that Leonardo had used this form (fig. 18.1) and that, as we have seen, it was a recurrent theme in sixteenth and seventeenth century practice and theory (fig. 18.2-4, pl. 55-56).

In the nineteenth century, drawing models of the regular solids was a regular part of the curriculum in art classes, the conviction being that these served as building blocks in learning how to render more complex shapes in nature.100 This purely academic exercise was transformed by the cubists into a new artistic movement, which deliberately reduced nature's complexities to cubes, pyramids and other regular solids. In the case of artists such as Paul Sérusier treatment of these basic forms tended to become almost an end in itself.

Interest in regular solids has by no means been limited to the cubists. In the past generation Lucio Saffaro has devoted great attention to these and related shapes. Trained as a physicist Saffaro has focussed on the potentials of geometric lines and colours in creating these forms.101 In cases such as the *Portrait of Kepler* (fig. 35.1), the effect is reminiscent of the Renaissance tradition. Frequently, however, as in his *Second Palladio* (fig. 35.2, cf.35.3-4), another concern is evident: creating figures which are simultaneously two and three dimensional, i.e. both flat and spatial in such a way that reading one part three dimensionally contradicts the surface realities of other parts of the painting. This concern with visual spaces that look fully realistic and yet are physically impossible is something he shared with his elder contemporary M.C. Escher.

Escher was fascinated by the regular solids. He claimed that they symbolize our desire for harmony and order. "But at the same time their perfection gives us a sense of helplessness. They are not inventions of the human spirit, because they existed as crystals in the earth's crust long before the advent of man."102 Escher used to have models of various regular and semi-regular solids on his desk. He was particularly interested in stellations. At the time he was convinced that this aspect of his work would probably have very little public appeal, but added that:

> nonetheless I am very much satisfied with it and if you ask me: Why do you make such mad things, such absolute objectivities, which no longer have anything personal to them, then I can only answer: I simply cannot leave it alone.103

Escher consciously drew on the past. One of his spirals104 drawn in 1953 (fig. 20.7), was a deliberate variation of the Florentine hat shape that Barbaro had made a leitmotif of his chapter headings (fig. 20.1-6). In addition to various three dimensional models105, he devoted at least a dozen of his drawings to these shapes.106 Already in 1943 in his *Reptiles* he had drawn lizards crawling over a regular dodecahedron. His rendering of complex polyhedra began in December 1947 with his *Crystal* (fig. 40.1) and with his study for *Stars of August* 1948 which he developed in October of that year. This adopted Leonardo's open version of an octahedron, but involved three such forms intertwined to produce a stellated effect. Traditionally the solids had been symbols of the elements and connected with static situations. Escher transformed this idea by making these solids into open cages for his lizard like animals (fig. 40.2), a theme which he developed in *Gravitation*, where a stellated dodecahedron housed twelve imaginary animals. Kepler had linked the solids with minerals and the natural world. Some of Escher's drawings implicitly followed this tradition. For instance, in *Opposites*, also termed *Order to Chaos* (fig. 40.3), he contrasted the orderly perfection of the polyhedral form with the chaotic by products of the man made world. In other cases Escher illustrated the reverse, as in his *Double Planetoid* where a chaotic natural landscape was inscribed within the orderly pyramidal forms of a man made world. Similarly, in his

Crystal he deliberately contrasted the regularity of his solid, a cube fitted into an octahedron, with the irregularity of the rocks in the background, an idea which he developed in *Order and Chaos II*.

Renaissance artists had hand painted their more complex models in order that one could distinguish more clearly their various intertwining layers. By contrast, Escher's use of colour tended to heighten the ambiguity of the reading, forcing the viewer to consider in turn a series of competing possibilities. This was particularly apparent in his *Stereometric Figure* of 1961, and reflected his different goals. In the Renaissance the geometric game was a very serious attempt to catalogue natural forms and their underlying laws of transformation. For Escher the geometric game was much more playful, a means of revealing competing interpretations and possible realities. In the everyday world, for instance, we are surrounded by square or rectangular buildings with corners at 90o to one another. In his *Flatworms* (fig. 40.4) Escher explored how tetrahedra and octahedra could be combined to create a coherent space with surfaces at 45o to one another. Escher also explored impossible images. Where Renaissance artists saw their task as carefully recording models of actual objects, Escher set about creating shapes such as his *Waterfall* which could not have a direct physical model. In all this Escher emphasized the craft dimension of his work and compared himself with a troubadour who constantly draws upon well known motifs which are then interwoven in new ways.107 While this emphasis on technique and systematic approach made him suspect in the eyes of some artists, it brought Escher fame from unexpected quarters such as crystallography and mathematics.

7. Visual Mathematics

One of the few twentieth century mathematicians who have emphasized the importance of geometry as an independent domain of thought not to be reduced to a simple branch of algebra is H.S.M. Coxeter. His own fascination with regular solids led to an important study of fifty-nine stellations of the icosahedron (fig. 43.1-3) 108 and a standard work on regular polytopes. His *Introduction to Geometry* 109 was much more than a simple restatement of Euclid's *Elements*. Where Euclid was concerned strictly with geometrical situations, Coxeter was concerned with what might be termed a dynamic transformational geometry. His inspiration drew partly from developments in botany110 and biology111 (cf. fig. 35.1-4). For instance, Church (1904)105 had demonstrated that leaf arrangements, technically termed phyllotaxis, on trees and plants followed specific mathematical rules, relating to the Fibonacci series mentioned earlier. Similarly, D'Arcy Wentworth Thompson had demonstrated that the spirals of a nautilus shell followed logarithmic rules.112

Coxeter also became interested in Escher113 partly because he helped to visualize some of the 17 basic types of symmetries114, partly because of his Möbius knots.115 In the 1860's the Viennese mathematician Möbius had drawn attention to curious properties of certain knots which raised basic questions on the meaning of inside and outside. His treatment of these Möbius knots had been purely mathematical. Escher became fascinated by them and drew a series of three-dimensional versions illustrating their properties. These in turn excited Coxeter116 who saw therein new ways of visualizing mathematical paradoxes, particularly those connected with topology, i.e. the study of unchanged or invariant mathematical properties of an object as it is stretched, twisted, bulged, folded, bent, etc. If these developments reveal a new interplay between art and mathematics, they also represent a new chapter in our story of geometrical games which becomes the more important when seen in the context of other mathematical developments.

Euclid had focussed his attention on the properties of triangles and other polygonal shapes in a flat two dimensional plane. In the nineteenth century when mathematicians such as Gauss, Riemann, Bolyai and Lobachevsky explored what happened if this flat two dimensional plane were replaced by spherical, parabolic on other surfaces, each alternative was hailed as a new non-Euclidean geometry.117 Topology has led us to recognize that such situations are in fact special cases of a larger mathematical system118; that it is not a question of rejecting Euclid but rather a challenge of seeing how his methods can be extended beyond simple flat dimensions.

The properties of this new dynamic geometry are so unlikely that we literally need to see them to begin to understand them. The soap bubble is an excellent example. It begins in those pipes that children use to blow bubbles as a two dimensional circular surface. When air is blown into the pipe the surface expands into a partial sphere, a half sphere, and then becomes ever more spherical until finally it is transformed into an independent sphere.119 The blowing of a bubble demonstrates in a few seconds the very transformations that Leonardo was trying to codify on his *Geometrical Game*: the difference being that he spent literally hundreds of pages trying to make the equivalents of snapshots of individual stages in this process using ruler and compass, whereas we can rely on formulae to provide the augmenting or diminishing iterations involved. Alternatively a motion film can be made of these processes and its motion can then be frozen in order to study specific instants in the development, as has been done in Emmers' film of experiments by Fred Almgren and Gene Taylor at Brown University120.

Quite independent of these developments, two German mathematicians, Wolf and Wolff took these ideas considerably further. They believed that calculus had provided a scientific foundation for understanding nature's functions and their goal was to find a corresponding foundation for morphology: not only nature's forms but those of art and architecture also.121 Their search was guided by the concept of symmetry which, in their definition, aims at understanding the necessary aspects of objective beauty122 and only becomes apparent through endless repetitions,--the buzz word is now iterations--of symmetrical operations. Their work began with a list of 13 different kinds of symmetries, and paid considerable attention to the regular solids both in isolation and as inscribed bodies. Haeckel (1899)123 had drawn attention to the importance of polyhedral and other symmetrical shapes in plants and minerals. Wolf and Wolff (1956) cited these, added numerous other examples (fig. 35.1-4) and drew evidence from an amazing range of sources including architectural ground plans, magnetic and electrical fields, molecular structures and chains, rose windows, ornaments on pottery, and the packing of atoms.

Meanwhile, Benoit B. Mandelbrot, another individual at the frontiers of research was apparently unaware of these efforts but interested in related problems. Where geometers such as Coxeter were exploring the potentials of mathematical symmetries and transformations in analysing nature's forms, Mandelbrot focussed attention on how standard geometry, with its lines, circles, cones and spheres was unable to describe the shape of clouds, mountains, coastlines and trees.124 He wanted to quantify difficult shapes which scientists had characterized in terms such as "grainy, hydralike, in between, pimply, pocky, ramified, seaweedy, strange, tangled, tortuous, wiggly, wispy and wrinkled"125 or as he put it "to investigate the morphology of the amorphous."126 His solution was to develop a new geometry of nature using irregular and fragmented shapes and patterns which he called fractals.

Mandelbrot deliberately took a stand against the non visualizing trends of the past centuries. He was convinced that mathematicians had "increasingly chosen to flee from nature by devising theories unrelated to anything we can see or feel."127 He lamented the near-total visual barrenness of Weierstrauss, Cantor and Peano, noting that this had become the case in physics since Laplace. For Mandelbrot:

> The wide and uncritical acceptance of this view has become destructive. In particular in the theory of fractals, 'to see is to believe...' Graphics is wonderful for matching models with reality.... A formula can relate to only a small aspect of the relationship between model and reality while the eye has enormous powers of integration and discrimination.... In addition graphics helps find new uses for existing models.128

This modern defence of visual model making methods of the Renaissance would be of interest in its own right. In the context of our story it becomes the more interesting because Mandelbrot's work inadvertently led him back to the regular solids. He was interested in random shapes and numbers. He began with a triangle. Repetitions generated a Sierpinski gasket (fig. 37.1). A three-dimensional version thereof produced a Sierpinski arrowhead (fig.37.2) consisting of tetrahedrons in stacks.129 Other iterations i.e. repetitions under slightly different conditions generated a Menger sponge (fig. 37.3) based on hexahedrons or cubes.130 The complexities of this story are eloquently recounted in Mandelbrot's manifesto, *The Fractal Nature of Geometry* (1983), and how all this links up with chaos theory and the latest developments of science has been made accessible to a general audience by Gleick's book.131 If the details of fractals cannot concern us here, the general trend helps us to understand why Renaissance methods of visualization have become important once again and why there is renewed fascination with regular and not so regular solids. They exemplify methods which mathematicians and scientists have just rediscovered as being essential to progress.

In a world where articles and books of five years ago are frequently considered obsolete, there is something sobering in the realization that artists three or four hundred years ago created images which can match the most daunting examples of computer graphics today. Nor is this only in the sense of a certain humility that comes in recognizing that the human spirit did not awaken for the first time in our own generation. I mean that understanding of earlier goals can help us to look at our own efforts more critically, as an example involving perspective and fractals may illustrate.

As noted earlier the fundamental principle underlying Renaissance perspective was that size and distance were inversely related, i.e. that if distance was doubled, size was halved etc. There was a tacit assumption that scale did not matter. Whether it was a big building or a little man everything obeyed the size distance rule. Or so it seemed. The advent of telescopes and microscopes provided ample demonstrations that the situation was not so simple but for over three centuries no one was troubled by the implications. Then in 1967 Mandelbrot132 wrote a three page paper about the length of the coastline of Britain noting how this was a function of the scale one used, i.e. it got longer as one's ruler became shorter because one had to take into account more twists and turns of the coastal landscape. By implication size and shape varied not only with distance but also with scale and any serious attempt at a science of forms would need to take into account that these rules of scaling would be different for every substance. Ironically, fractals (fig. 39.1) which were designed to solve these problems continued the very assumptions they had undermined by relying on patterns independent of scale just as perspective did. As a result while fractals generate captivating and often very beautiful symmetries, they produce fractal leaves (fig. 39.2) which do not record real leaves in the way that careful drawings or photographs using perspective do. To be sure enthusiasts such as Barnsley try to tell us differently:

> Fractal geometry will make you see everything differently.... It can be used to make precise models of visual structures from ferns to galaxies. Fractal geometry is a new language. Once you can speak it, you can describe the shape of a cloud as precisely as an architect can describe a house.133

The fact that existing fractals, beautiful though they be, fall far short of these claims (e.g. fig. 41.1-3) has led some critics to reject them entirely and insist instead on the importance of perspectival photographs in recording nature. As is so often the case, however, rather than either-or, it is almost certainly a question of both perspective and fractals, or some modification of both. For what was originally intended as the solution has thus far only brought into focus a deeper problem. We now know that size and shape are functions of both distance and scale. But the challenge of rendering nature's regularities and irregularities remains, and will no doubt inspire new geometric games in the generations to come.

CONCLUSIONS

Our story began with the Babylonians where regular and semi-regular objects were used both practically for weights and measures and ornamentally as jewellry. Their cosmological significance emerged in the fourth century B.C. with Pythagoras and led to a series of writings by Theaetetus, Hippasus, Plato, Euclid, Apollonius, Hysicles and Archimedes. It was Archimedes who dealt systematically with the thirteen semi-regular solids and, we are told, first referred to mechanics as "geometry at play."1 In a sense therefore he is the founder of geometric games and it is interesting to note that Mandelbrot2 also considers him the founder of midpoint displacement used in fractals, our latest version of geometric games. On the surface the Mediaeval contribution was largely one of transmission and commentary. Pappus recorded the contributions of Archimedes, Isidorus, the architect of Hagia Sophia added a fifteenth book to Euclid's *Elements*. In the Arabic tradition Ishaq ibn Hunain commented on problems of the regular solids, while Averroes brought these back into the context of cosmological debates of Aristotle vs. Plato. But in a subtle way the mediaeval tradition changed the whole nature of the discussions. Mathematicians such as Proclus spread the idea that the whole of Euclid's *Elements* was devoted to understanding the regular solids. At the same time Christian commentators from Boethius onwards assumed that geometry was literally measurement of the earth. What had been mainly a metaphysical problem in Antiquity was now implicitly also a physical problem.

We found that this physical dimension was made explicit by four fifteenth century individuals, Regiomontanus, Piero della Francesca, Luca Pacioli (fig. 42.1) and Leonardo da Vinci who explored cosmological, theological and scientific aspects of the regular solids, and introduced the idea of model making as a preparatory stage to representation. This practical aspect of model construction was taken further by the Nürnberg artists Dürer and Hirschvogel in the first half of the sixteenth century before being reintegrated into metaphysical schemes by the goldsmiths Jamnitzer (fig. 42.2), Lencker(fig. 42.3) and artists such as Stoer and Neudorffer (fig. 42.4). We traced how this three-dimensional perspectival metaphysics produced some of the most unlikely images in the history of the human imagination. Italians such as Barbaro, Sirigatti and Vasari Jr. popularized some of these images, but focussed attention on variants of stellated spheres and cylindrical hat shapes rather than expanding the repertory. Others such as Danti praised the idea of drawing and transforming regular geometrical shapes perspectivally but gave no examples. In spite of publications it is striking that a clearly cumulative pattern does not emerge (cf. Appendix I). Indeed by 1600 interest in representing the solids was on the decline. The first half of the seventeenth century saw one major contribution by Halt and some attempts at revival by Nicéron and the Jesuit Father Dubreuil but these were increasingly exceptions.

Why this happened was the theme of part two. One reason was the development of perspectival and other instruments. Initially these were designed as aids in representing the solids. But they also led to measuring their relative proportions. Once these were known the need to actually draw the solids diminished. We found that the proportional compass and sector were integrally connected with these developments. A second reason was bound up with developments in the subject-object distinction by Kepler and Descartes. Study of the regular solids now proceeded on two quite different paths. One treated them physically, assumed that these regular and semi-regular shapes were inherent in minerals and other objects of the natural world, and led eventually to the development of crystallography as an independent science. A second approach treated them mentally as mathematical abstractions which led via Euler, Poinsot, and Cauchy to polyhedra becoming a specific field of mathematical enquiry. The final part of our survey summarized developments during the past three hundred years, tracing factors that led to a renewed interest in visualization and model making.

Viewed in global terms a series of stages can be identified. In Antiquity the regular solids were seen in geometrical terms, as being linked with the world of ideas, symbolic of nature but not directly linked with it. During the Renaissance there emerged a process of model making which served as a go between linking concrete nature and abstract mathematics. The early seventeenth century brought two approaches. One assumed that it was no longer necessary to think of the solids as intermediary models because they were somehow part of nature's own structure. The other analyzed the solids mathematically in terms of relative proportions, which could then be translated into algebraic formulae. This algebraic analysis eventually gained ascendancy to the point that visualization was considered an inferior method of communication. With Lagrange, this approach became respectable in physics. Even so models continued to hold their fascination in science until the rise of quantum physics with Bohr and Born in the early 1920's, when they were abandoned by mainstream scientists.

Meanwhile in artistic circles models had remained a basic aspect of teaching in the academies ever since the sixteenth century. Models of the regular solids in particular were a usual feature of nineteenth century artistic education. With the cubists in the early twentieth century the regular solids acquired a new significance. Rather then merely being linked with preparatory exercises, they now provided a key to discerning fundamental structures of objects underlying their constantly changing appearances. Artists such as Escher remained fascinated by regular and semi-regular models for different reasons. Art now became an explanation of changes and transformations of regular geometrical shapes.

This approach captured the imagination of mathematicians such as Coxeter (fig. 43.1-3) and Ernst3 who were concerned with a new kind of dynamic mathematics involving, for instance, the topological properties of soaps suds changing into soap bubbles. Their commitment to understanding changing forms in nature led to renewed interest in model making and visualization generally. Escher's work also led him to explore seemingly regular geometrical shapes for which no corresponding physical model could exist. This too was of interest to mathematicians who were also concerned with visualizing problems for which no obvious models existed or could exist in nature. Meanwhile Mandelbrot had set himself an even more imposing task of finding a geometry that would deal with all of nature's irregularities ranging from coastlines and mountain-sides to the changing shapes of clouds. This quest led him to recognize that abstract formulae on their own were not enough, that various kinds of simulation using models were necessary and that visualization was therefore a basic requirement.

The story we have outlined suggests that the whole history of early modern science may nee to be reconsidered. The great advances in astronomy and quantum physics in the twentieth century led many historians to assume that the scientific revolution took place in astronomy (Copernicus) and physics (Galileo) and that these breakthroughs led to a triumph of algebraic methods where progress was measured by complexity of formulae. Our story suggests that a number of other fields played a central role in the emergence of early modern science, including anatomy, mechanics, geology, geography, botany and zoology, namely precisely those which are described in earlier classification systems as the descriptive sciences of nature (*beschreibende Naturwissenschaften* as opposed to *mathematische Wissenschaften*) In which case the development of modern science needs to be seen simultaneously as a triumph of visual methods and abstract results. Indeed where earlier solutions involved oppositions between ideal and real, concrete and abstract, geometry and arithmetic, algebra and diagrams, we need to recognize that evolution is embracing not replacing, that all these approaches have their uses, and that it is more fruitful to look at these various solutions on a continuum linking concrete nature and abstract mathematics. For Archimedes geometry at play was limited to mechanics. For Roger Bacon and other mediaeval philosophers the geometrical game was extended from the realm of man made machines to all regular shapes in nature, but it remained primarily an activity of God which man could merely imitate. In a sense we are still doing that. Except now examples of the past are challenging us to create even richer geometric games in the future and in a mysterious way the search for nature's outer forms is also an exploration of the inner contours of the imagination which are unending.

LEONARDO DA VINCI (Based on)

(1452-1519)

During the Renaissance serious interest in the regular solids began with Regiomontanus and Piero della Francesca. Leonardo da Vinci (1452-1519) pursued these themes. While at the court of Milan he produced a set of illustrations for Luca Pacioli's book *On Divine Proportion* (1496-1499, published 1509). These were copied in a manuscript now in the Ambrosian Library in Milan. In addition to the regular solids and variants such as a stellated dodecahedron (pl.3), this presented a twenty six-sided rhombicuboctahedron (pl.1), a seventy two-sided figure plus twenty one columnar and pyramidal shapes. The twenty six-sided figure recurs in a contemporary portrait of Luca Pacioli (pl.3). Friar Giovanni of Verona used a number of these forms as motifs in inlaid wood in both Monte Oliveto near Siena (pl.4.6) and in Santa Maria in Organo in his hometown of Verona (pl.7). Leonardo also made his own studies of the subject. Sketches in the *Codice Atlantico* in the Ambrosian Library confirm that he was familiar with the five regular and the thirteen semi regular Archimedeian solids a century before Kepler who is often credited as being the first to study these forms systematically.

WENZEL JAMNITZER

(1508-1585)

Born in Vienna, Jamnitzer became a citizen of Nürnberg in 1534 and went on to become one of the most famous goldsmiths in Europe. He is best known for a treatise on *Perspective of Regular Bodies*, which he sent as a manuscript to Archduke Ferdinand in Innsbruck in 1557 and published in 1568. Preparatory drawings for these are now in Berlin (e.g.pl. 8). Jamnitzer linked the five regular solids with the five vowels (a,e,i,o,u), giving 24 variants of each shape (pl.9-13). These were followed by open versions of the solids again linked with the vowels (e.g. pl.14-15), variants on the seventy two-sided figure and related spherical shapes (pl.16-17), plus a series of pyramids (pl.18-19) and cylinders (e.g. pl.20). This book was reprinted in 1617 with pirate editions in 1608, 1618, and 1626. Jamnitzer also wrote two scientific treatises, one formerly at Dresden, lost in the war, the other now in the Victoria and Albert Museum in London.

HANS LENCKER
(152_-1585)

Trained as a goldsmith, Lencker became a citizen of Nürnberg in 1551 and later played a role in town politics. In 1567 he published *Perspective of Letters* in which he presented all the letters of the alphabet in perspective (pl.21-22). This was followed by a series of eight semi-regular shapes (e.g. pl. 23-25). In 1571 Lencker published a second treatise entitled Perspective which focussed on perspectival instruments for recording the regular solids. His technical renown took him to the court of Saxony in Dresden and also brought commissions from the courts at Kassell and Munich. A second edition of his *Perspective of Letters* (1596) contained a few additional geometrical shapes.

ANONYMOUS MASTER

(fl.1565-1600)

This anonymous manuscript now at the Herzog August Bibliothek in Wolfenbüttel contains 36 hand painted folios. Many of these are variants of forms from Jamnitzer's *Perspective of Regular Bodies* of 1568 and the work has been attributed to him. This is unlikely for several reasons. The drawings, while beautiful lack the precision of line characteristic of Jamnitzer. Second, precarious shapes such as the pile of 13 cylinder-like shapes (pl. 31) would not have been in keeping with Jamnitzer's metaphysical concerns which led him to favour highly symmetrical objects. Third, someone as inventive as Jamnitzer would not simply have copied a form based on his rival Lencker (pl. 30) without improving on it. It is noteworthy that the more original shapes in this series (e.g. pl. 27, 32) are adorned with little animals again with a playfulness that is out of keeping with Jamnitzer's metaphysics. However, similar animals recur in an anonymous cabinet produced in Augsburg in 1566 (pl. 57-58). Were both perhaps made in the same workshop in Augsburg?

LORENZ STOER
(c.1537-c.1621)

Virtually nothing is known of the early life of this master of painting and drawing except that he grew up in Nürnberg where he remained until 24 May 1557. His first --and it was usually thought his only-- principal work was *Geometry and Perspective*, finished in 1556 and published in 1567, which contained a series of 11 regular and semi-regular solids balanced on one another in a setting of architectural ruins and fanciful motifs (pl. 33-36). These decorative designs were intended as models for inlaid wood. In 1557 Stoer moved to Augsburg where he composed two further manuscripts. One, now at the Herzog August Bibliothek in Wolfenbüttel, contains a series of 32 illustrated folios in no apparent order (e.g pl. 37-40). The other, now in the University Library at Munich, contains a series of 360 hand-painted drawings systematically arranged in what is probably the most beautiful manuscript on perspective ever produced (e.g. pl. 42-51). It is likely that a drawing by Stoer now in the Graphic Collections in Munich (pl.41) was one of his alternative designs for a title page to one of these works. As in the anonymous Wolfenbüttel manuscript there is again a dog in the background. Were such animals the signature of Stoer's workshop?

PAUL PFINZING VON HENFENFELD

(1554-1599)

Paul Pfinzing came from a Patrician family in Nürnberg where he was active as a painter and engraver and from 1587 onwards was active in city politics. In the last two years of his life he prepared *A Beautiful Short Extract of Geometry and Perspective*. Although he referred briefly to Italian and French authors he focussed on developments in Nürnberg, from the time of Dürer through to his own day. Pfinzing illustrated various instruments used in drawing the regular solids and some of the related shapes that resulted. His original intention was to distribute the book privately to friends, which explains why almost every one of the handful of remaining copies varies slightly in terms of content and is handpainted (pl.52-56). His book was reprinted posthumously under the title of *Optics* in 1616. Pfinzing was also the author of *Geometrical Method, that is a Short Well Founded and Thorough Treatise on Land Surveying and Measurement* (1598) and a *Perpetual Calendar* (1623).

MASTERS OF INLAID WOOD

(c.1566-1600)

The hand painted versions of the solilds produced by Jamnitzer and Lencker in Nürnberg and Stoer in Augsburg (e.g. pl. 52) inspired some of the most remarkable examples of inlaid wood furniture ever produced. The three pieces that follow are all by anonymous master craftsmen. The first dated around 1570 is a reading desk (pl. 53) produced at Nürnberg now in the Museum for Decorative Arts in Frankfurt. The second now in the Museum of Applied Arts in Cologne (pl. 54-55) is a cabinet with numerous drawers each revealing new scenes. The third, (pl. 56-56) is a cabinet dated 1566 which was produced in Augsburg and is now in the Westphalian Museum in Münster. The last two illustrations in this section show details of the sides of that cabinet.

LORENZO SIRIGATTI

(fl.1590-1596)

Very little is known about Cavalier Lorenzo Sirigatti. As an architect he is known for the Palazzo Salvetti (Via Ghibellina 73-76) in Florence. The manuscript version of his treatise on the *Practice of Perspective* lacks many of the illustrations that occur in his published work of 1596 which is in two parts. Part one illustrates elementary principles of perspective. Part two focusses on regular solids, spheres and *mazzocchio* shapes --originally a type of Florentine hat--in various combinations (e.g. pl. 59-63). He is said to have taught Giorgio Vasari, Jr. A second edition of Sirigatti's work appeared in 1625.

PETER HALT

(fl.1620-1653)

Nothing is known of Peter Halt's early life. He was active as an architect, stone mason and engraver in Schorndorf. He was also a publisher in Augsburg and had his treatise on *Perspectival Drawing* printed there in 1625. In the tradition of Jamnitzer, Halt linked the five regular solids with the five vowels on his title page. Halt's 175 illustrations were also influenced by Lorenz Stoer who was still active in Augsburg in 1621 and may therefore have known him personally. While often imaginative and interesting his drawings lack the polished quality of his predecessors. There is a record of his still being active in Ulm in 1653 but we know of no further books by him.

CHRISTOPH ANDREAS NILSON

(fl.1812)

In the eighteenth and nineteenth centuries the regular and semi-regular solids became topics of mineralogy and mathematics. Treatises on perspective usually dealt only in passing with the regular solids. Probably the last treatise in which the solids played a dominant role was Nilson's *Introduction to Linear Perspective or a Thorough Instruction in Perspectival Stereometry* published in Augsburg and Leipzig in 1812, the year of Tchaikovsky's overture. Nilson's atlas contained a series of engravings showing both regular and irregular solids in garden settings and application used previously by Kirby in his popularization of Brook Taylor's work (1763). Most of the actual shapes were simple adaptations of sixteenth century examples

NOTES

INTRODUCTION

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4. H. M. Cundy and A. P. Rollett, *Mathematical Models,* Oxford: Clarendon Press, 1951.

5. Magnus J. Wenninger, *Polyhedron Models*, London: Cambridge University Press, 1971.

6. H.S.M. Coxeter, *Regular Polytopes*, New York: Macmillan, 1963 (2nd ed.).

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CHAPTER ONE. COSMOLOGY, THEOLOGY AND MATHEMATICS

1. Otto Neugebauer, *The Exact Sciences in Antiquity*, New York: Dover Publications, 1969, pp. 46-47.

2. F. Lindemann, "Zur Geschichte der Polyeder und der Zahlzeichen," *Sitzungsberichte der mathematisch-physikalischen Classe der k. b. Akademie der Wissenschaften zu Munchen*, Band XXVI, Jahrgang 1896, Munchen: Verlag der K. Akademie, 1897, pp. 625-757, particularly pp. 635, 645-649.

3. Ibid., p. 725. Cf. Carl Fridrich Naumann, *Elemente der Mineralogie*, 8th ed., Leipzig: Engelmann, 1871, pp. 20, 569.

4. Ibid., pp. 629-631, based on an article by Herrn Geheimrath Conze, "Uber ein Bronzgeräth in Dodekaederform," *Westdeutsche Zeitschrift für Geschichte und Kunst*, Trier, Jahrgang XI, 1892, pp. 204ff.

5. Ibid., pp. 686 ff.

6. Ibid., pp. 729-735.

7. For an introduction to these debates see *The Thirteen Books of Euclid's Elements*, ed. T.L. Heath, New York: Dover, 1956, vol. 3, p. 438. For a more detailed treatment see the article by

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8. See Euclid's *Elements*, ed. Heath, as in note 7, vol. 2, pp. 97-100.

9. On this topic, see H. E. Huntley, *The Divine Proportion. A Study in Mathematical Beauty*, New York: Dover Publications, 1970.

10. See Lindemann, as in note 2, p. 729. Cf. Heath, as in note 7, vol. 3, p. 438.

11. Plato, Timaeus, trans. H.D.P. Lee, Harmondsworth: Penguin, 1965, pp. 72-85, particularly p. 75.

12. See Heath as in note 7, vol. 3, p. 438. For details see the other works listed in note 7.

13. Ibid.

14. A Greek manuscript of Pappus is now in the Vatican Library, *Ms. Vat. Gr. 218*. For a standard edition see: *Pappus Alexandrinus, Collectionis quae supersunt*, ed. F. Hultsch, Berlin: Apud Weidmannus, 1876, vol. 1, pp. 133-163. Cf. E. M. Bruins, "The Icosahedron from Heron to Pappus", *Janus*, Leiden, 1957, vol. 46. pp. 173-183.

15. Heath, as in note 7, vol. 1, pp. 115-116.

16. Heath, as in note 7, vol. 13, p. 439.

17. Ibid., vol. 1, pp. 5-6, vol. 3, p. 512.

18. Ibid., pp. 512-519.

19. Aristotle, *The Works of Aristotle*, ed. David Ross, Oxford: Clarendon Press, 1930, vol. 2; De caelo, 306b, 4-9, 307b 19.

20. See Marshall Clagett, *Archimedes in the Middle Ages*, Madison: University of Wisconsin Press, 1964-1980, 4 vol. Of particular relevance is volume three: *The Fate of the Medieval Archimedes, 1300 to 1565. Part Three. The Medieval Archimedes in the Renaissance, 1450-1565. Memoirs of the American Philosophical Society*, Vol. 125, Part B). This contains sections on Regiomontanus (pp342-383); Piero della Francesca (pp. 383-415, particularly 386, 398-406 re: solids); Luca Pacioli (pp. 416-461, particularly 455-458 re: solids); and Leonardo da Vinci in relation to Archimedes.

21. Hero of Alexandria, *Metrica*, ed. E.M. Bruins, Leiden: E.J. Brill, 1964 (*Textus Minores*, vol. XXXV).

22. As in note 14.

23. Lindemann, as in note 2, p. 640. This interpretation is debated.

24. Ibid., p. 636.

25. Ibid., p. 636, Cf. Tischler, "Ostpreussische Gräberfelder," *Schriften der physikalisch-Ökonomischen Gesellschaft zu Königsberg*, Königsberg, Jg. 19, 1878, pp. 239 ff.

26. Heath, as in note 7, vol. 3, pp. 519-520.

27. Heath, as in note 7, vol. 1, pp. 75-76.

28. Ibid., vol. 1, p. 78.

29. Ibid.

30. Erich Auerbach, *Mimesis, The Representation of Reality in Western Literature,* trans. Willard R. Trask, Princeton: Princeton University Press, 1953.

31. Boethius, *De institutione arithmetica...Geometria*,.ed. G. Friedlein, Leipzig: B.G. Teubner, 1867.

32. Important in this respect was Al Farabi's *On the sciences* which became known to the West through Gundisallinus and Grosseteste. For an introduction to this literature see: A. C. Crombie, *Robert Grosseteste and the Origins of Experimental Science*, Oxford: Clarendon Press, [1962]. For specialized discussions see: Heinrich Suter, "Die Abhandlung des Abu Kamil Shoga b. Aslam über das Fünfeck und Zehneck", *Biblioteca Mathematica*, Leipzig, 3 Folge, Bd. 10, Heft 1, 1910, pp. 15-42.and Jan P. Hogendijk, "Greek and Arabic Constructions of the Regular Heptagon", *Archive for History of Exact Sciences*, Berlin, vol. 30, No.3/4, 1984, 197-330.

33. See for instance Roger Bacon, *Speculum mathematica*, Frankfurt: Wolfgang Richteri, sumptibus Antonij Hummij, 1614.

34. For an introduction to the mathematical context of the fifteenth century see Paul Lawrence Rose, *The Italian Renaissance of Mathematics*, Geneva: Librairie Droz, 1975 (*Travaux d'humanisme et renaissance*, CXLV).

35. *De quinque corporibus aequilateris, quae vulgo regularia nuncupantur, quae videlicet eorum locum impleant naturalem et quae non contra commentatorem Aristotelis Averroem.* This title is cited in J.G. Doppelmayr, *Historische Nachricht von den Nürnbergischen Mathematicis*, Nürnberg, 1739, p. 19.

36. Although the original work has been lost its contents have been summarized in Regiomontanus, *Commensorator*, ed. Wilhelm Blaschke, Günther Schoppe, Wiesbaden: Verlag der Akademie der Wissenschaften und der Literatur in Mainz, 1956, particularly pp. 472-521. (*Akademie der Wissenschaften und der Literatur. Abhandlungen den Mathematisch-Naturwissenschaftlichen Klasse*, Jahrgang 1956, Nr. 7).

37. Ibid., p. 480: "Wenn man durch andere Teilung der Seite jeder Basis Fortschreitet, so erzeugt man unbegrenzte regelmässige Korper."

38. The *Commensurator* as cited in note 35.

39. Doppelmayr as in note 34.

40. Margaret Daly Davis, *Piero della Francesca's Mathematical Treatises*, Ravenna: Longo Editore, 1975.

41. See Luigi Vagnetti, "Considerazioni sui Ludi Matematici," *Studi e Documenti di Architettura*, Florence: Teorema, no. 1, 1972, pp. 173-259.

42. Piero della Francesca, *De prospectiva pingendi*, ed. G. Nicco-Fasola, Con una lettura di Eugenio Battisti, Florence: Le Lettere, 1984. A new edition is being prepared by Marisa Dalai Emiliani.

43. Ibid., p. 210 (fol. 82v, fig. LXXVIII).

44. For a discussion of this see the author's: *Leonardo Studies I*, as in note 2 of introduction, pp. 146-149.

45. Cf. W. R. Lethaby, *Architecture, Mysticism and Myth*, London: Percival and Co. 1892, pp. 254-272. Cf. Isa Ragusa, "The Egg Reopened," *Art Bulletin*, New York, vol. 53, 1971, pp. 435-443.

46. Daly Davis, as in note 40, pp. 119-120.

47. See, for instance: Piero della Francesca, "L'opera De corporibus regularibus di Pietro Franceschi detto Della Francesca, usurpata da Fra Luca Pacioli", ed. Girolamo Mancini, *Memorie della Real Accademia di Lincei*, Rome, serie V, vol. XIV, fasc. VIIB, 1916, pp. 441-580. Cf. Gino Arrighi, "Piero della Francesca e Luca Pacioli. Rassegna della questione del plagio e nuove valutazioni," *Atti della Fondazione Giorgio Ronchi*, Florence, vol. 23, 1968, pp. 613-625.

48. Plato, as in note 11, p. 74.

49. See: Leonardo da Vinci, *Manuscript F*, fol. 27v. For a discussion of this see the author's *Leonardo Studies I*, p. 171.

50. See Luca Pacioli, *Divina Proportione*, ed. Constantin Winterberg, Vienna: Carl Graeser, 1888. Reprint: Hildesheim: Georg Olms Verlag, 1974 (*Quellenschriften für Kunstgeschichte und Kunsttechnik des Mittelalters und der Neuzeit*, Neue Folge, II, Band), p. 35: "Propter admirari ceperunt philosophari."

51. Ibid., p. 35: "Quod nihil est in intellectu quin prius sit in sensu."

52. Ibid., pp. 40-41.

53. Ibid., pp. 43-44.

54. Ibid., p. 44: “E poi medianti sti a infiniti altri corpi detti dependenti."

55. Cf. Byrna Rackusin, "The Architectural Theory of Luca Pacioli: De Divina Proportione, Chapter 54," *Bibliothèque d'Humanisme et Renaissance,* Geneva, vol. 39, 1977, pp. 479-503 with 3 fig.

56. This passage has been cited and translated in the author's: *Sources of Perspective*, Munich: Saur, 1990.

57. For a full discussion of these see the author's: *Leonardo Studies I*, pp. 170-187.

58. Ibid., pp. 67-86.

59. Ibid., pp. 270-277.

60. Ibid., pp. 187-201.

61. Leonardo da Vinci, *Codex Forster I*, London, Victoria and Albert Museum.

62. Ibid., fol. 124-11v.

63. Leonardo da Vinci, CA 128ra (353r, c. 1508). In this and future references *CA* refers to *Codice Atlantico* in Milan. The first number is the pagination of the 1st edition (1896-1904). The second in brackets is the new pagination of Professor Augusto Marinoni.

64. E.g. CA 160rb (432r, 1515-1516), CA 167rb (455r, c. 1515) or CA 242rb (660r, c. 1517-1518).

65. E.g. CA 184vc (505v, c. 1516) and CA 174v (476v, c. 1517-1518). See on this problem: James Edward McCabe, *Leonardo da Vinci's De ludo geometrico,* Ph.D., University of California at Los Angeles, 1972.

66. CA 99vb (273br, c. 1515).

67. CA 170ra (463v, c. 1516).

68. CA 45va (124v, 1515-1516).

69. These themes are developed in the author's: *Structure and Method in the Notebooks of Leonardo da Vinci*, Brescia, 1991.

70. For an excellent introduction to this problem see: Kenneth D. Keele, *Leonardo da Vinci's Elements of the Science of Man*, New York: Academic Press, 1983, particularly chapter 4. See also: *Leonardo Studies I*, pt. II, chapter 3.

71. CA 151ra (407r, c. 1500).

72. Ms. K 49 [48 et 15]r.

73. CA 130va (360r, c. 1517-1518).

74. For another discussion see the author's "Visualisation and Perspective", *Leonardo e l'età della ragione,* ed. Enrico Bellone e Paolo Rossi, Milan: Scientia 1980, pp. 185-210.

75. These were noted briefly by Daly-Davis, as in note 40, p. 73 but were analysed in detail by Marisa Dalai-Emiliani," I poliedri Platonici....", *Lettura Vinciana*, 1988, Florence: Giunti, 1988.

76. Cf. Luciano Rognini, *Le Tarsie di Santa Maria in Organo*, Vicenza: Arte Grafiche delle Venezie, 1978. (Monumenti di cultura e d'arte Veronesi a cura della Banca Popolare di Verona).

77. Albrecht Dürer, *Underweysung der Messung*, Nürnberg, 1525, fol. Miiiv-viv.

78. Ibid., Mviv:

> Auch sind noch vill hubscher corpora zumachen/die auch in einer holen kugel mit all iren ecken an ruren/aber sie haben ungleyche felder/der selben wil ich eins teyls hernach auf reyssen/und gantz aufgethan/auf des sie ein netlicher selbs zamen mug legen/welicher sie aber machen will der reyss sie grosser auf ein zwifach gepasst papier/un schneyd mit einem scharpfen messer auf der einen seyten all ryss durch den einen pogem papiers/und so dan all ding auss dem ubrigen papier geledigt wirt/als dan legt man das corpus zusamen/so lest es sich geren in den rissen piegen/darumb nym des nachfolgeten auf reyssens acht dan solche ding sind zu vill sachen nutz.

79. Ibid., Nvr:

So man des for gemachten corpora mit glatten schnitten jre eck weg nimbt/und dan die beleybenden eck/aber hinweg nymbt/so mag man manicherley corpora darauss machen.

Auss dissen dingen mag man gar manicherley machen/so jr teyl aufeinider versetzt wirt/das zu dem aushauen der zeulen und jren zirden dinet.

80. E.g. Latin: *Quatuor his suarum institutionum geometricarum libris* (Paris, 1532, 1534, 1557); French: *Les quatres livres*... (Paris, 1557).

81. Cf. Albrecht Dürer, *The Complete Drawings of Albrecht Dürer*, ed. Walter L. Strauss, New York: Abaris Books, 1964, vol. 6, pp. 2839-2882.

82. Anonymous, *Eyn schon nützlich Buchlein*, ed. Hieronymus Rodler, Siemeren: Rodler, 1531.

83. Augustin Hirschvogel, Geometria. *Das Buch Geometria ist mein Namen. All Freye Kunst aus mir zum ersten kamen. Ich bring Architectura und Perspectiva zusamen*, Nürnberg, 1543

84. Heinrich Lautensack, *Des Circkels und Richtscheyts auch der Perspectiva...Underweisung*, Frankfurt: Georg Raben, 1564.

85. Two important exhibition catalogues provide the best introduction to Jamnitzer's complex work: Michael Mathias Prechtl and Elisabeth Rücker, ed., *Jamnitzer, Lencker, Stoer, Drei Nürnberger Konstruktivisten des 16, Jahrhunderts*, Nürnberg: Albrecht Dürer Gesellschaft e.V. 1969 (Katalog 11); Klaus Pechstein, Ralf Schürer, und Martin Ungerer, *Wenzel Jamnitzer und die Nürnberger Goldschmiedekunst 1500-1700. Eine Ausstellung im Germanischen Nationalmuseum*, Nürnberg von 28 Juni-15 September, 1985, München: Klinckhardt und Biermann, 1985.

86. In 1556 he was made "Genannter des grösseren Stadtrates;" in 1564 he became "städtischen Hamptmann" and in 1573 he continued as "Genannten des kleineren Stadtrates."

87. *Perspectiva corporum regularium. Das ist/ein fleyssige Fürweysung/wie die Fünff Regulirten Corper/darvon Plato inn Timaeo/Und Enclides inn sein Elementis schreibt/etc. Durch einen sonderlichen/newen/behenden und gerechten Weg/der vor nie in gebrauch ist gesehen worden/gar künstlich inn die Perspectiva gebracht/und darzu ein schöne Anleytung/wie auss denselbigen Fünff Cörpern one Endt gar viel andere Corper/mancherley Art und gestalt/gemacht/unnd gefunden werden mügen*... Nürnberg; 1568.

88. These and other analogies were explored by Albert Flocon in his introduction to the 1964 French edition (Paris: Alain Brieux). These were translated by Ulla Gostomski and re-arranged by Dr. Gerard Mammel as "Albert-Flocons Jamnitzer Interpretation" in the 1969 catalogue, as in note 85.

89. Published anonymously in a pirate edition as *Sintagma in quo varia eximaque corporum diagrammata ex praescriptio opticae exibentu*r, Amsterdam: J. Jansson 1608 with reprints in 1618 and 1626.

90. Hans Lencker, *Perspectiva literaria...Das ist...wie man alle Buchstaben des gantzen Alphabets...in die Perspectif einer flachen Ebnen bringen mag*, Nürnberg: Lencker, 1567.

91. Albrecht Dürer, *Underweysung der Messung*, Nürnberg, 1525.

92. Geoffroy Tory, *Champ fleury*, Paris; Petit Pont a l'enseigne du Pot Cass,, 1529.

93. Cf. Flocon's introduction to the 1964 edition and the 1969 catalogue, p. 13, as in note 85 above.

94. Ibid., p. 13: "wie man ohne Vokale nicht sprechen könne, ohne die regulären Körper nichts in der perspektivischen Reisskunst erreiche."

95. Lorenz Stoer, *Geometria et Perspectiva. Hier inn etliche zerbrochene Gebaew, den Schreiner. In eingelegter Arbeit dienstlich*, Augsburg: Michael Manger, 1567. The work is said to have been written in 1556, i.e. the year before he left Nürnberg to live in Augsburg.

96. [Lorenz Stoer], listed in Wolfenbüttel, Herzog August Bibliothek, under call no.: *Cod. 47.1 Aug. quart.*

97. [Lorenz Stoer], *Die fünff corpora regularia auff viel und mancherley Arth und Weiss zerschnitten*, Munich, Universitätsbibliothek under call no. *20 Cod. Ms. 592*.

98. Ibid., "Geometria et perspectiva corpora," fol. 14-210.

99. Ibid., "Volgen allerley triangel und Krentz perspectivisch," 1599, fol. 211-219.

100. Ibid., "Geometria et perspectiva," fol. 220-247.

101. Ibid., "Volgen allerley perspectivische Stuckh doppelt drie und vierfach ufeinander," fol. 248-266.

102. Ibid., "Geometria et perspectiva corpora regulata et irregulata," fol. 267.

103. Paul Pfintzing, *Soli Deo Gloria. Ein schöner kurtzer Extract der Geometriae unnd Perspectivae*, Nürnberg: Valentin Fuhrmann, 1599.

104. Cf. Wolfenbüttel, Herzog August Bibliothek, *Cod. Guelf, 415.1. Nov.* and *Cod. Guelf. 78 Extrav.2o*.

105. One of the Bamberg copies is handpainted, i.e. *Ms. Ma.F.5*.

106. Lucas Brunn, *Praxis perspectivae, das ist von Verzeichnungen ein aussfuhrlicher Bericht darinnen das jenige was die Scenographi erfordert begrieffen wird*, Nürnberg, Leipzig: Simon Halbmeyer, gedrucket Lorentz Kober.

107. Hans Lencker, *Perspectiva darin ein leichter Weg allerley Ding in Grund zu legen durch ein sonderlichen Instrument gezeigt wird*, Augsburg: Michelspacher, 1615; Ulm: J. Meder, in Verlegung F. Michelspachers, 1616.

108. Paul Pfintzing, *Optica, das ist gründtlich doch kurtze Anzeigung*, ed. Stephan Michelspacher, Augsburg: David Francken, Steffan Michelspacher, 1616.

109. Lorenz Stoer, *Geometria et perspectiva*, Augsburg, 1617.

110. Peter Halt, *Perspectivische Reiss-Kunst*, Augsburg: Dan. Franckhen, 1625.

111. Daniele Barbaro, *La pratica della perspettiva*, Venice: B. e R. Borgominieri, 1568.

112. Ibid., pp. 64-67.

113. Ibid., p. 104.

114. The two manuscripts are in Venice, Biblioteca Marziana with the call no. *Ms. It. 40 (5447)* and *Ms. It. IV, 39 (5446).*

115. This is *La practica della prospettiva*, Marciana, *Ms. It. IV, 39 (5446).*

116. In a sense this paradox is evident from the outset of perspective. Brunelleschi's discovery involved an instrument. We have descriptions of his instruments but no exmaples of his art. Alberti claimed perspective was not possible without instruments. Leonardo's drawing showed a window used to record an armillary sphere (fig. 25.1). We have no armillary sphere drawn so accurately on its own by Leonardo. The problem continues through the sixteenth century. The machines for rendering were often rendered more vividly than the objects they were intended to represent.

117. Giacomo Barozzi Vignola, *Le due regole della prospettiva pratica*, ed. R.P.M. Egnatio Danti, Rome: Francesco Zanetti, 1583.

118. Ibid.

119. Ibid., probl XI, Prop. XL, Annotazione:

> Since, beyond the description of rectilinear figures it is very useful for the Perspectivist to know how to transmute one figure into another, I wish in these three following paragraphs to show the normal way not only to transmute a circle or any other rectilinear figure that is wished into another, but also how to expand and diminish it into any proportion that is described in order that in this book the Perspectivist will have all that which is required for such a noble practice.
>
> Translation by the author of:
> Perchè oltre alla descrizione delle figure rettilinee, apporta gran commodità al Prospettivo il saperle trammutare d'una nell'altra, ho voluto in queste tre seguenti Proposizioni mostrare il secondo modo la via comune non solamente di trammutare il circolo e qual si voglia figura rettilinea in un altra, ma anco di accrescerle, e diminuirle in qual si voglia certa proporzione, accio in questo libro il Prospettivo abbia tutto quello , che a cosi nobil pratica fa mestiere.

120. Lorenzo Sirigatti, *La Pratica di prospettiva*, Venice: Girolamo Franceschi Sanese, 1596.

121. Giorgio Vasari, il giovane, *Prospettiva del Cavalier Giorgio Vasari*, Florence, Uffizi, Archivio del Gabinetto des Disegni, *Mss. Nr. 4945-4885 e 5986-5045*.

122. Pietro Accolti, *Lo inganno degl'occhi*, Florence: Pietro Cecconcelli, 1625.

123. Ibid., p. 58: "sara sempre necessario primieramente il far fabbricare e metere insieme, o di legname, o cartone, o altra materia, quell'obbietto che il pittore si propone rappresentare in veduta di prospettiva."

124. Ibid., p. 74: “Onde mi e parso inventar alcuna maniera, perche possa ciascuno, nella brevita de spatij di sue tele, o dentro gl'angusti termini di sua stanza, conseguire ciascuna sudetta operazione.”

125. Concerning this work see the excellent study by P. M. Sanders, "Charles de Bovelles Treatise on the Regular Polygons," *Annals of Science*, London, vol. 41, 1984, pp. 513-566.

126. Oronce Finé, *De geometria practica sive de practicis longitudinum, planorum & solidorum: hoc est linearum, superficierum, & corporum mensionibus aliijsque mechanicis es demonstratis Euclidis elementis corollarius. Ubi et de quadrato geometrico et virgis seu baculis mensorijs*, Strasbourg: Knoblochiana, 1544.

127. Ibid., p. 72: Ut polygonae multilateraeque figurae sub mensuram cadant.

128. Ibid., p. 122: De caeterorum regularium corporum dimensione.

129. Oronce Finé, *Quadratura circuli...De mensura circuli....De multangulorum omnium...De invenienda...Planisphaerium*, Paris: Apud Simon Colinaeum. 1544.

130. Ibid., pp. 41-71: "De absoluta rectilinearum omnium et multangulorum figurarum (quae regulares adpellantur) descriptione, tam intra quam extra datum circulum, ac super quavis oblata linea recta. Libellus hactenus desideratus."

131. The book was Oronce Finé's *De rebus mathematicis hactenus desideratis libri iii*, Paris: Ex officina Michaelis Vascosani, which contained pp. 95v-130v: "Corollarium de regularium polygonorum descriptione in dato circulo per isoscela triangula.”

132. Jean Cousin, *Livre de perspective*, Paris: Le Royer, 1560.

133. Claude de Boissière Delphinois, *L'art d'arithmétique contenant toute dimension, tant pour l'art militaire que par la géométrie et autres calculations, revue et augment, par Lucas Tremblay parisien professeur des Mathematiques*, Paris: par Guillaume Cavellat, 1561.

134. Ibid., 53r: Règle générale de la quantité de tous vaisseaux.

135. Ibid., 72v: Ioinct aussi que son but estoit de faire un abregé de l'arithmétique et de la g,om,trie a celle fin de ioindre les quantitez discrettes avecques les continues.

136. For a standard discussion of anamorphosis see Jurgis Baltrusaitis, *Anamorphoses Les perspectives dépravées*, Paris: Flammarion, 1984. There is a less complete English version: *Anamorphic art*, Cambridge: Chadwick Healy, 1977.

137. Jean Francois Nicéron, *La perspective curieuse ou magic artificielle des effets merveilleux de l'optique, de la catoptrique de la dioptrique*, Paris: P. Billaine, 1638.

138. Jean Francois Nicéron, *Thaumaturgus opticus*, Paris: Francois Langlois, 1642 with a reprint 1663.

139. Jean Dubreuil, *La perspective pratique*, Paris: Melchior Tavernier, 1642-1649, 3 vol.

CHAPTER TWO. CRYSTALLOGRAPHY, MATHEMATICS AND ART

1. Archimedes, *Opera omnia*, ed. J.L. Heiberg, Leipzig: B. G. Teubner, 1910-1975.
For an analysis of this work see Marshall Clagett, *Archimedes in the Middle Ages*, Madison: University of Wisconsin Press, 1964-1980, 4 vol. Cf. above note 20.

2. Hero of Alexandria, *Geometricorum et stereometricorum reliquiae*, ed. F. Hultsch, Berlin: B. G. Teubner, 1865.

3. Leonardo Pisano, *Il liber abbaci di Leonardo Pisano*, ed. Baldassare Boncompagni, Rome: Tipografia delle scienze matematiche e fisiche, 1867.

4. Luca Pacioli, *De Divina Proportione*, as in note 50 above, pp. 92-95.

5. Ibid., pp. 106-109: "Del modo a mesurare tutte sorte colonne e prima delle rotonde."

6. Ibid., pp. 106-107.

7. Ibid., p.116 : "Onde de dicti regulari non mi cavo altramente qui extenderme per haverne gia composto particular tractato alo illustrissimo affine de vostra Ducale celsitudine Guido ubaldo opera a Sua Signoria dicata." Cf. Clagett, as in note 20, vol.3, part III, p.455.

8. Oronce Finé, "De absoluta rectilinearum...," as in note 129 above.

9. This had been dealt with earlier in the anonymous, *Breve corso di matematica*, Modena, Biblioteca Estense, *Ms. It. 211 = a.W.6.22*.

10. Giacomo Barozzi da Vignola, *Le due regole della prospettiva pratica*, Rome: Zanetti, 1583,

11. There is an example in the Adler Planetarium in Chicago. For two examples by others see: Anthony Turner, *Scientific Instruments*, London: Sotheby's Publications, 1987, pl. 59, 143.

12. Fabrizio i Gaspare Mordente, *Il compasso*, Antwerp: Plantin 1584, p.60.

13. Ibid., Preface, p.*3v.

14. This instrument has been described by Stillman Drake, "Galileo and the First Mechanical Computing Device," *Scientific American*, New York, vol. 234, no. 4, April 1976, pp. 104-133, particularly p. 107.

15. See the Modena manuscript, as in note 9, fol. 58r.

16. Giovanni Paolo Gallucci, *Della fabrica et uso di diversi stromenti di astronomia et cosmografia*, Venice, 1598.

17. For an excellent survey of this literature see: Menso Folkerts, "Die Entwicklung und Bedeutung der Visierkunst als Beispiel der praktischen Mathematik der fruhen Neuzeit," *Humanismus und Technik*, Berlin, Bd. 18, Heft 1, 1974, pp. 1-41.

18. For an introduction to this literature see the author's: "Mesure, quantification et science," in: *L'Epoque de la Renaissance, 1400-1600*, Budapest: Akademiai Kiado, vol. 4, (in press).

19. There is an example of such a compass in the Science Museum (London).

20. See Leonardo da Vinci, *Codice Atlantico*, fol. 157vb (425v), 358ra (1064r), 248ra (672r).

21. Wilhelm IV, Landgraf Hessen, (attributed to), *Circini proportionalis descriptio*, Biblioteca apostolica Vaticana, *Reg. lat. 1149*, fol. 2r:

1. Datam rectam lineam iuxta datam proportionem dividere
2. Datam lineam circularem in propositas partes secare
3. Datam superficiem in similem superficiem multiplicare aut minuere
4. Datam corpus in simile corpus multiplicare aut minuere
5. Rationem cuiuslibet diametri ad suam circumferentiam invenire
6. Superficiem circularem aut quadratam in aliam transferre
7. Datum globum et quinque corpora regularia in sese invicem transferre.

22. See, for instance, L. von Mackensen, *Die erste Sternwarte Europas mit ihren Instrumenten und Uhren, 400 Jahre Jost Bürgi in Kassel*, Munchen: Callwey Verlag, 1982, pp. 89-114.

23. Ibid., pp. 128-129.

24. Hans Lencker, Perspective, as in note above.

25. Levinus Hulsius, *Dritter Tractat der mechanischen Instrumenten Levini Hulsii, Beschreibung und Unterricht dess Jobst Burgi Proportional Circkels*... Frankfurt: Levini Hulsii, 1604.

26. Philip Horcher, *Libri tres in quibus primo constructio circini proportionum edocetur*, Mainz: Apud Balthasarum Lippium 1605.

27. Galileo Galilei, *Le operazione del compasso geometrico e militare*, Padua: Casa dell'Autore, Per Pietro Marinelli 1606.

28. This will be the subject of the author's *Mastery of Quantity*.

29. Luca Pacioli, as in note 50 above, p.18:

> Cinque corpi in natura son prodocti
> Da naturali semplici chiamati
> .....
> Quale Platone vol che figurati
> Lesser dien a infiniti fructi
> Ma perche elvacuo la natura aborre
> Aristotil in quel de celo e mundo
> Per se non figurati volsse porre.
> Pero lingegno geometra profondo
> Di plato edeuclide piacque exporre
> Cinqualtri che in spera volgan tundo
> Regolari daspeto iocundo.
> Comme vedi delati e base pare.
> E unaltro sexto mai sepo formare.

30. Leonardo da Vinci, *Ms. F* 27v.

31. Johann Doppelmayr, as in note 35 above, p. 19:
“Es scheinet dass Jamnitzer auch Lencker ihre perspectivische Vorstellungen mit Farben illuminiret und dann solche öffters in Tabulas striatas disponiret.”

32. Gerard Mammel, ed., "Albert Flocons Jamnitzer Interpretation," as in note 88 and 85 above, p. 9:
“Diese Geritzte Tafeln sind wahrscheinlich verzerende Platten mit parallelen Plättchen die eine zweite und dritte Platte bilden, wenn man von der Seite auf die Objekte blickt."

33. Ibid., p. 9:
...wie man in der Optik von Risner lesen kann: Nürnberger Goldschmiede haben eine wunderbare Vorrichtung geschaffen, die die Kunst der Perspektive oder Szenographie auf das glücklichste vollendet und deren bemaltes inneres so vollkommen angeordnet ist in seiner Genauigkeit und seiner Verkürzungen, dass den Anblick Halluzinationen erweckt.

34. Johann Kepler, *L'étrenne ou la neige sexangulaire*, tr. Robert Halleux, Paris: Vrin 1975, p. 66.

35. *Oxford English Dictionary,* Second Edition, Oxford: Clarendon Press, 1989, vol.XVI, pp. 896-897.

36. Johannes Kepler, *Prodromus dissertationum continens...Mysterium Cosmographicum*, Tübingen, 1596.

37. Johannes Kepler, *Harmonices mundi*, Linz: Sumpt. Godefridi Tampachii, excudebat Ioannes Plancus, 1619.

38. Johannes Kepler, *Strena seu de nive sexangula*, Frankfurt: apud Godefridum Tampach, 1611.

39. Ibid., as in note 34, p. 62.

40. Ibid., p. 73.

41. Ibid., p. 65.

42. Ibid., p. 74.

43. Ibid., p. 74:

elle ne tend pas seulement a produire des corps naturels mais encore elle s'amuse d'ordinaire a des jeux relachés, ce qui ressort de nombreux exemples de mineraux. La raison de tout cela, moi je la reporte du jeu (nous disons qui la nature joue) a une intention serieuse.

44. Ibid., p. 74.

45. Ibid., p. 74, Cf. p. 104.

46. Ibid., p. 81.

47. Ibid., p. 81:

Mais la faculté formatrice de la terre ne se limite pas a une seule figure, elle connait toute la géométrie et y est exercée.

48. Ibid., p. 81.

49. Ibid., p. 81 where he refers to: Johannes Bauhinus, *Historia novi et admirabilis fontis balneique Bollensis*....Montbliart, 1598.

50. Ibid., p. 81.

51. Ibid., pp. 109-137 contains an essay by Robert Halleux, "De la strena de Kepler a la naissance de la cristallographie."

52. Cf. *Greek Mathematical Works I. Thales to Euclid*, tr. Ivor Thomas, London: Heinemann, 1967, (*Loeb Classical Library* vol. 335), pp. 86-100.

53. Cf. Cecil J. Schneer, "The Renaissance Background to Crystallography" *American Scientist*, Champaigne, vol. 71, 1983, pp. 254-263.

54. Descartes, *Principes*, ed. Charles Adam et Paul Tannery, Paris: Leopold Cerf, (*Oeuvres de Descartes, vol. 12*), 1904, p. 64 :

Quand je ne recois point de principes in Physique qui ne soient aussi receus en Mathematique, afin de prouver par démonstration tout ce que j'en deduiray, a que ces principes suffisent d'autant que tous les phainomenes de la nature peuvent estre expliquez par leur moyen.

55. For a brief summary see Schneer, as in note 53, pp. 260-261. For a standard work see: I. I. Schafranovskij, *Kristalograficzeskie predstavlenija. I. Keplera i ego traktat, "O Sestingolom snege,"* Moscow, 1971.

56. Nikolaus Steno, *De solido intra solidum naturaliter contento dissertationis prodromus*. Florence: Ex typographiae sub signo stellae, 1669.

57. Domenico Guglielmini, *Riflessioni filosofiche dedotte dalle figure de sali*, Bologna, 1688.

58. A.M.A. Cappeller, *Prodromus crystallographiae sive de crystallis proprie sic dictis, commentarium*, Luzern, 1723.

59. Abraham Gottlob Werner, *Von der Äusserlichen Kennzeichen der Fossilien*, Leipzig,1774.

60. Jean Baptiste Louis Rom, de Lisle, *Essai de cristallographie*, Paris: Didot, 1772, cf. Paris: Impr. de Monsieur, 1783, 4 vol.

61. Torbern Bergman, *Commentatio de tubo ferrumentario, ejusdemque usu in explorandis corporibus praesertim mineralibus*, Vienna: Apud J. P. Kraus, 1779, *Sciagraphia regni mineralis secundum principia proxima digesti*, Leipzig: In bibliopolis J. G. Mülleriano, 1782. .

62. Georges Louis Leclerc, Comte de Buffon, *Époques de la nature* (1779) which is volume 6 of his *Histoire naturelle générale et particulière*, Paris, 1774-1779.

63. Ren, Just Haüy, *Traité de minéralogie*, Paris: Louis, 1801.

64. See Barbara Keyser, "Between Science and Craft, The Case of Berthollet," *Annals of Science*, London, vol. 1990, pp.

65. Cf. John Sinkakas, *Mineralogy*, New York: Von Nostrand, 1964, p. 11.

66. William Lawrence Bragg, *The Crystalline State*, London: G. Bell and Sons Ltd. 1933 and *Atomic Structure of Minerals*, Ithaca: Cornell University Press, 1937.

67. E.g. F. Donald Bloss, *Crystallography and Crystal Chemistry, an Introduction*, New York: Holt, Rinehart and Winston, 1971, Cf. Boris K. Vainshtein, *Modern Crystallography In Symmetry of Crystals, Methods of Structural Crystallography* New York: Springer Verlag, 1981.

68. See Sinkakas, as in note 65, pp. 114, 551-552:
Isometric
Tetragonal
Hexagonal

Orthorhombic
Monoclinic
Triclinic

69. H. S. M. Coxeter, *Introduction to Geometry*, New York: John Wiley and Sons, 1961, p. 413.

70. Cf. note 67 above.

71. *International Tables for Crystallography*, vol. 4, *Space-Group Symmetry*, ed. Theo Hahn, Dordrecht: D. Reidel Publ. Co. 1983.

72. Euclid, *Elementa geometrica, libris XV....His accessit decimus sextus liber, de solidorum regularium sibi invicem inscriptorum collationibus, tum etiam coeptum opusculum de compositis regularibus solidis plane peragendum*, ed. Franciscus Flussate Candalla, Paris: apud Ioannem Royerum, 1556.

73. Euclid, *Euclidis posteriores libri sex a X and XV. Accessit XVI de solidorum regularium compilatione.* Auctore Chritophoro Clavio. Rome: apud Vincentium Accoltum, 1574.

74. There were further editions in 1589 (Rome: Apud Bartholomaeum Grassium), 1591 (Cologne: Expensis Ioh. Baptistae Ciotti), 1603 (Rome: A. Zannettum), 1607 (Cologne: Apud Gosvinum Cholinum), 1607 (Frankfurt: Ex officina typographica N. Hofmanni, sumptibus I. Rhodij) and 1654 (Frankfurt: J. Rosae).

75. Euclid, *Euclidis elementorum geometricorum libro tredecim Isidorum et Hypsiclem et recentiores de corporibus regularibus.* Antwerp: Ex officina H. Verdussii, 1645 which contains pp. 513-532: "Commentarius in Franciscum Flussatem Candallam de quinque solidis regularibus."

76. Euclid, *Euclidis beginselen der meetkonst, vervaat in 15 boeken, naar by 't 16 boek* Fr. Flussatis Candallae. Amsterdam: Johannes van Keulen, 1695.

77. Euclid, *Elements of Geometry. In XV Books. With a Supplement of Divers Propositions and Corollaries.* London: Printed by R & W. Leybourn for G. Sawbridge 1660-1661. There were further editions of this in 1714 and 1722 (London: Printed...W. Redmayne), 1732 (London: Printed for Daniel Midwinter) and 1751 (London: Printed for W. and J. Mount and T. Page).

78. Euclid, *Elementa libris XV....Accessit decimussextus liber, de solidorum regularium sibi invicem inscriptorum collationibus. Novissime collati sunt decimus septimus et decimus octavus, de componendorum, inscribendorum et conferendorum compositorum solidorum inventi, ordine et numero absoluti,.* Auctore D. Francisco Candalla. Paris: Apud Iacobum Du Pays, 1578.

Brückner, as in note 3, p. 156 credits Foix with introducing two new bodies, the exoctaedron with 6 square and 8 triangles and the icosidodecahedron with 20 triangles and 12 pentagons.

79. Michael Stifel, *Arithmetica integra*, Nürnberg, 1543. Cf. Nürnberg: Apud Johan Petreium, 1544.

80. Rafael Bombelli, *L'algebra, Ms. B, 1569, Biblioteca dell'Archiginnasio di Bologna*, ed. Ettore Bortolotti, Bologna: Nicola Zanichelli, 1929, pp. 279-302. Also important in this context was Simon Stevin concerning whom see E. J. Dijksterhuis, *Simon Stevin*, Hague: Nijhoff, 1970, p.44. Here it is noted that truncating planes may pass:

a) through the mid points of the sides meeting in a vertex.
b) through the points dividing these sides in the ratio 1:2 so that the lesser segment is adjacent to the vertex.
c) as in b) but so that the lesser segment has to the greater the same ratio which a side of a face has to the sum of the side and a diagonal.

81. Adrianus Romanus, *Ideae mathematicae pars prima, sive methodus polygonorum qua laterum perimetrorum a arearum cujuscunque polygoni investigandoram ratio exactissima et certissima una cum circuli quadratura continentur,* Antwerp: Apud Ioannem Keerbergium, 1598.

82. Cf. T. Kaori Kitao, "Imago and Pictura: Perspective, Camera Obscura and Kepler's Optics," in *La prospettiva rinascimentale. Codificazioni e trasgressioni*, ed. Marisa Dalai Emiliani, Florence: Centro Di, 1980, pp. 499-510.

83. See, for instance, Gilbert Ryle, *The Concept of Mind,* Harmondsworth: Penguin, 1949.

84. René Descartes, "Mathematica de solidorum elementis excerpta ex manuscriptis Cartesii," in: *Oeuvres de Descartes*, ed. Charles Adam et Paul Tannery, Paris: Leopold Cerf, 1908, vol. 10, pp. 265-276.

85. Cf. Brückner, as in note 3 of Introduction, pp. 58-60. See also L. Euler, "Elementa doctrinae solidorum; Demonstratio nonnullarum insignium proprietatum quibus solida hedris planis inclusa sunt praedita," *Novi Commentariae Academiae scientiarum imperialis Petropolitanae*, Petrograd (i.e. Leningrad), tom IV (ad annum 1752 et 1753), 1758, pp. 109 ff, pp. 140ff.

86. Louis Poinsot, "Mémoire sur les polygones," *Journal de l'école polytechnique*, Paris, 10 cahier, tom. IV, 1810, pp. 16-46.

87. Baron Augustin Louis Cauchy, "Recherches sur les polyèdres," *Journal de l'école polytechnique*, Paris, 16 cahier, tom. , 1812, p. 69.

88. Simon-Antoine-Jean Lhuilier, "Mémoire sur les solides réguliers," *Annales de mathématiques pures et appliquées (Gergonnes Annalen),* Nismes, Paris vol. III, 1812-1813, p. 233, Cf. the same author's *Polygonométrie ou de la mesure des figures rectilignes et abrégé d'isopérimétrie élémentaire ou de la dépendance mutuelle des grandeurs et des limites des figures,* Geneva: Bard, Manget et Cie., 1789.

89. The Science Museum in London has a number of models illustrating his methods.

90. Felix Klein, *Vergleichende Betrachtungen über neuere geometrischer Forschungen*, Erlangen: Deichert, 1872 (*Programm zum Eintritt in die philosophische Fakultät und den Senat der K.*

Friedrich-Alexanders Universit„t zu Erlangen). For an English translation see: "A comparative review of some researches in geometry," *Bulletin of the American Mathematical Society,* New York, 1893, vol. 2, pp. 215-249.

91. Felix Klein, *Vorlesungen über das Ikosaheder und die Auflösung der Gleichungen vom fünften Grade*, Leipzig: B. G. Teubner, 1884. This is also available in English as: *Lectures on the Ikosahedron and the Solution of Equations of the Fifth Degree*, trans. G.G. Morrice, London: Trubner, 1883.

92. Brückner as in note 3 of Introduction.

93. David Hilbert, S. Cohn-Vossen, *Anschauliche Geometrie*. Berlin: J. Springer, 1932. Translated as: *Geometry and the Imagination*, tr. P. Nemenyi, New York: Chelsea Publishing Co., 1952.

94. Josephe Louis, Comte Lagrange, *Méchanique analytique*, Paris: Chez La Veuve Desaint, 1788. Cf. Dirk J. Struik, *A concise history of mathematics,* New York: Dover, 1948, p. 134.

95. Otto Löwe, *Ueber die regulären und Poinsot'schen Körper und ihre Inhaltsbestimmung vermittelst Determinanten*, Marburg: Druck der Universitäts-Buchdruckerei, 1875.

96. Karl Georg Christian von Staudt, *Beiträge zur Geometrie der Lage*, Nürnberg: Fr. Korn, [1847?].

97. Bartel Leendert Van der Waerden, *Einführung in die algebraische Geometrie*, Berlin: J. Springer, 1939.

98. Edwin Abbott, *Flatland: A Romance of Many Dimensions by a Square,* London: Seeley and Co., 1884.

99. For an introduction to this complex literature see Linda Dalrymple Henderson, *The Fourth Dimension and Non-Euclidean Geometry in Modern Art*, Princeton: Princeton University Press, 1983.

100. For an introduction to these materials see the author's: *Sources of Perspective,* Munich: Saur, 1990, pp. 122-130.

101. See for instance Lucio Saffaro, *Saffaro Grafica e pittura*, ed. Marisa Dalai Emiliani, Sergio Marinelli, Verona: Museo di Castelvecchio, 1979.

102. Cited in *De werelden van M.C. Escher,* ed. J.L. Locher, C.H.A. Broos, M.C. Escher, G.W. Locher, H.S.M. Coxeter, Amsterdam: Meulenhoff, 1971, p. 53:

> Zij symbolisieren op weergaloze wijze ons verlangen naar harmonie en orde, maar tevens geeft hun perfectie ons een gevoel van hulpeloosheid.... Zij zijn geen uitvindingen van de menselijke geest, want zij waren als cristallen lang voor de mens in de korst van onze aarde aanwezig.

103. Cf. *Leben und Werk M.C. Escher*, ed. J.L. Locher, Eltville: Rheingauer Verlagsgesellschaft, 1986, p. 146:

> Freilich ein Bild, welches in nur geringem Ausmass an den Geschmack des Publikums appelliert und sich denn auch wahrscheinlich sehr schlecht verkaufen lässt. Selbst bin ich jedoch sehr zufrieden damit, und wenn Du mich fragst: Warum machst Du solche verrückten Sachen, derartige absolute Objektivitäten, die nichts persönliches mehr an sich haben, dann kann ich nur antworten: Ich kann es einfach nicht lassen.

104. Cf. Ibid., pp. 71, 146.

105. Cf. Ibid., pp. 72, 146.

106. These are, using the numbers of the above catalogue, in chronological order:

327	March	1943	*Reptiles*
353	December	1947	*Crystal*
358	August	1948	*Study for Stars*
359	October	1948	*Stars*
365	December	1949	*Double Planetoid*
366	February	1950	*Opposites (Order and Chaos I)*
380	June	1952	*Gravitation*
395	April	1954	*Four Surfaced Planet*
402	August	1955	*Order and Chaos II*
431	January	1959	*Flat Worms*
438	May	1961	*Stereometric Figure*
439	October	1961	*Waterfall*

107. *Leben und Werk M.C. Escher*, as in note 96, p. 155.

108. H. S. M. Coxeter, as in note 7 of Introduction.

109. H. S. M. Coxeter, *Introduction to Geometry*, New York: John Wiley and Sons, 1961.

110. Ibid., pp. 169-172.

111. Ibid., pp. 125-126.

112. A. H. Church, *The Relation of Phyllotaxis to Mechanical Laws*, London: Williams and Norgate, 1904, p. 1

113. D'Arcy Wentworth Thompson, *On Growth and Form*, Cambridge: Cambridge University Press, 1917.

114. H.S.M. Coxeter, as in note 102, pp. 57-59.

115. Ibid., p. 413.

116. H.S.M. Coxeter, as in note 95, p. 53.

117. Ibid., pp. 53ff. Cf. H.S.M. Coxeter, "The non-Euclidean Symmetry of Escher's Picture Circle Limit III," *Leonardo*, Oxford, vol. 12, 1979, pp. 19-25.

118. For an introduction see Linda Dalrymple Henderson, as in note 92, pp. 3-10.

119. For an interesting example see Michael Barnsley, *Fractals Everywhere*, Boston: Academic Press, 1988, p. 9, where a plane triangle is related to its equivalent on a spherical surface.

120. For more specialized studies by Frderick J. Almgren see*: Plateau's Problem. An Invitation to Varifold Geometry*, New York: W. J. Benjamin, 1966, and *Existence and Regularity almost Everywhere of Solutions to Elliptic Variational Problems with Constraints*, Providence: American Mathematical Society, 1976 (*Memoirs of the American Mathematical Society*, vol. 4, no. 165).

121. Michele Emmers (Rome) has made a series of films on dynamic visual mathematics.

122. K. Lothar Wolf und Robert Wolff, *Symmetrie. Versuch einer Anweisung zu gestalthaften Sehen und sinnvollen Gestalten systematisch dargestellt*, Münster: Böhlau Verlag, 1956, Vorbemerkung:

> Das vorliegende Buch ist das erste abschliessende Ergebnis langer Bemühungen fur die Morphologie eine entsprechende exakte Grundlage du gewinnen, wie sie die mit den funktionellen Geschehen in der Natur verbundenen Erscheinungen seit langen in der Infinitesimalrechnung gefunden haben.

123. Ibid., pp. 1-3: "Das objectif Schöne oder Freiheit und Notwendigkeit."

124. Ernst Haeckel, *Kunstformen der Natur*, Leipzig: Bibl. Institute, 1899. Cf. his *Art Forms in Nature*, New York: Dover, 1974.

125. Benoit B. Mandelbrot, *The Fractal Geometry of Nature*, New York: W. H. Freeman and Co., 1983 ed., p. 1.

126. Ibid., p. 5.

127. Ibid., p. 1.

128. Ibid.

129. Ibid., pp. 21-22.

130. Ibid., pp. 140-143.

131. Ibid., pp. 144-145.

132. James Gleick, *Chaos. The Birth of a New Science*, New York: Viking, 1987.

133. Benoit B. Mandelbrot, "How long is the Coast of Britain? Statistical Self-Similarity and Fractional Dimension," *Science*, London, vol. 155, 1967, pp. 636-638. In his *Fractal Geometry of Nature*, as in note 118, p. 28, Mandelbrot notes a precedent in Steinhaus, 1954.

134. Michael Barnsley, *Fractals Everywhere*, Boston: Academic Press, 1988, p. 1.

CONCLUSION

1. Plutarch, "Marcellus" in: *Plutarch's Lives*, ed. Arthur Hugh Clough, London: Dent, 1910, vol. 1, p.471.

2. Cf. Heinz-Otto Peitgen, Dietmar Saupe, ed., *The Science of Fractal Images*, New York: Springer Verlag, 1988 contains a foreword by Benoit Mandelbrot, p. 11.

3. Bruno Ernst in his essay in *Leben und Werk M.C. Escher*, as in note 96, p. 135 listed seven reasons why Escher was interesting for mathematicians:

   1. Integration of different worlds
   2. Illusion of space
   3. Regular divisions of surfaces
   4. Perspectives
   5. Regular Bodies and Spirals
   6. The Impossible
   7. The Unending

APPENDIX I

Name	Sides	Contents		
REGULAR (PLATONIC) SOLIDS				
1. Tetrahedron (Pyramid)	4	4T		
2. Hexahedron (Cube)	6	6S		
3. Octahedron	8	8T		
4. Icosahedron	12	12P		
5. Dodecahedron	20	20T		
SEMI-REGULAR (ARCHIMEDEIAN) SOLIDS				
1. Truncated Tetrahedron	8	4T	4H	
2. Cuboctahedron	14	8T	6S	
3. Truncated Octahedron	14	6S	8H	
4. Truncated Hexahedron	14	8T	6H	
5. Rhombicuboctahedron	26	8T	18S	
6. Rhombitruncated Cuboctahedron	26	12S	8H	6O
7. Icosidodecahedron	32	20T	12P	
8. Truncated Icosahedron	32	12P	20H	
9. Truncated Dodecahedron	32	12T	12D	
10.Snub Hexahedron (Cube)	38	32T	6S	
11 Rhombicosidodecahedron	62	20T	30S	12P
12.Rhombitruncated Icosidodecahedron	62	30S	20H	12P
13.Snub Dodecahedron	92	80T	12P	

	Piero	Pacioli	Dürer	Stevin	Bombelli	Colombichio
Euclid						
1	x	x	x	x	x	x
2	x	x	x	x	x	x
3	x	x	x	x	x	x
4	x	x	x	x	x	x
5	x	x	x	x	x	x
Archimedes						
1.	x	x	x	x	x	x
2	x	x	x	x	x	x
3	x	x	x	x		x
4	x		x	x	x	x
5	x	x	x			x
6						x
7		x		x		x
8	x	x				x
9	x			x		x
10			x			x
11						x
12						x
13						x

I. During the Renaissance Leonardo da Vinci was the first to study all the five regular and 13 semi-regular solids. This was also done by Canciano Colombichio in his *Nuovo trattato delle radici quadrate et cubiche* (Trieste, Biblioteca civica, R.P. Ms.2-33). If the above list reveals that Leonardo's work had no direct impact it also reveals that even printed works did not affect the major theoreticians of the time.

II. Leonardo da Vinci's studies of regular and irregular solids, with Kepler's names and modern equivalents.

APPENDIX III

Augustin Hirschvogel, *Geometry*, Nürnberg, 1543.

1. Rhombicosidodecahedron
2. Tetrahedron
3. Octahedron
4. Icosahedron
5. Hexahedron (Cube)
6. Dodecahedron
7. Truncated Tetrahedron
8. Icosidodecahedron
9. Truncated Hexahedron
10. Cuboctahedron
11. Truncated Octahedron
12. Rhombicuboctahedron

Daniele Barbaro, *La pratica della perspettiva*, Venice: Borgominieri, 1568

Name		Sides	Contents			
1. Tetrahedron (Pyramid)		4	4T			
2. Hexahedron (Cube)		6	6S			
3. Octahedron		8	8T			
4. Icosahedron		12	12P			
5. Dodecahedron	20	20T				
6. Truncated Tetrahedron	8	4T	4H			
7. Cuboctahedron	14	8T	6S			
8. Truncated Hexahedron	14	8T	6H			
9. Rhombicuboctahedron	26	8T	18S			
10. Truncated Octahedron	14	6S	8H			
11. Icosidodecahedron	32	20T	12P			
12. Truncated Dodecahedron		32	12T	12D		
13. Truncated Icosahedron	32	12P	20H			
14. Rhombitruncated Cuboctahedron		26	12S	8H	6O	
15. ----		38	24T	6S	8H	
16.Rhombitruncated Icosidodecahedron	62	30S	20H	12P		
17. ----		92	60T	12P	20H	
18. ----		92	12P	80H		
19. ----		92	60T	12D	20TW	
20. ----		60	30S	20H	10D	
21. ----		38	24T	6O	8TW	
22. ----		182	60T	90S	12P	20H
23. ----		56	24T	24S	8H	
24. ----		74	36S	24H	6O	8TW
25. ----		12	12T			
26. ----		24	24T			
27. ----		24	24T			
28. ----		60	60T			
29. ----		60	60T			
30. ----		48	48T			
31. ----		96	96T			
32. ----		120	120T			
33. ----		18	6S	12H		
34. ----		64	24T	6S	8H	
35. ----		12	24T			
36. ----		20				

III. Two further treatments of the regular and semi regular solids in perspective treatises of the sixteenth century which again follow a different pattern.

APPENDIX IV

Piero della Francesca, *Booklet of the Abacus*			Pseudo-Euclid, Elements, Book XV (i.e.Isidorus)
1. Octahedron	in a	Tetrahedron	2
2. Tetrahedron	in a	Hexahedron	1
3. Octahedron	in a	Hexahedron	3
4. Icosahedron	in a	Hexahedron	-
5. Hexahedron	in a	Octahedron	4
6. Tetrahedron	in a	Octahedron	5
7. Hexahedron	in a	Dodecahedron	8
8. Tetrahedron	in a	Dodecahedron	10
9. Octahedron	in a	Dodecahedron	9
10.Icosahedron	in a	Dodecahedron	7
11.Hexahedron	in a	Icosahedron	11
12.Tetrahedron	in a	Icosahedron	12
13.Dodecahedron	in a	Icosahedron	6

François Foix, *Elements Book XV*

1. Tetrahedron	in a	Hexahedron	1
2. Octahedron	in a	Tetrahedron	2
3. Octahedron	in a	Hexahedron	3
4. Hexahedron	in a	Octahedron	4
5.Dodecahedron	in a	Icosahedron	6
6. Tetrahedron	in a	Octahedron	5
7.Icosahedron	in a	Dodecahedron	7
8. Hexahedron	in a	Dodecahedron	8
9. Octahedron	in a	Dodecahedron	9
10. Tetrahedron	in a	Dodecahedron	10
11.Hexahedron	in a	Icosahedron	11
12.Tetrahedron	in a	Icosahedron	12
13. Icosahedron	in a	Hexahedron	-
14. Icosahedron	in a	Hexahedron	-
15. Octahedron	in a	Icosahedron	-
16.Icosahedron	in a	Octahedron	-
17.Dodecahedron	in a	Octahedron	-
18.Hexahedron	in a	Tetrahedron	-
19.Icosahedron	in a	Tetrahedron	-
20.Dodecahedron	in a	Tetrahedron	-
21.Any Reg. Solid	in a	Sphere	-

IV. Study of the regular solids led to careful exploration of how they could be inscribed within one another. Isidorus the architect of Hagia Sophia in Instanbul was among the first to do this sytematically in the fiteenth book of the Elements often attributed to Euclid himself. During the Renaisance both Piero della Francesca and Fran‡ois Foix developed their own versions.

APPENDIX V

V. A table from René Descartes treatise on regular solids illustrating his abstract treatment of the problem.

INDEX OF ILLUSTRATIONS

Fig. 12.2, Ibid. fol. 27v.

Fig. 12.3, Ibid. fol. 27r.

Fig. 12.4, Ibid. fol. 26v.

Fig. 13.1, Ibid. fol. 26r.

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